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$D0$-$D4$ brane tachyon condensation to a BPS state and its excitation spectrum in noncommutative super Yang-Mills theory

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Abstract: We investigate the $D0$-$D4$-brane system for different B-field backgrounds including the small instanton singularity in noncommutative SYM theory. We discuss the excitation spectrum of the unstable state as well as for the BPS $D0$-$D4$ bound state. We compute the tachyon potential which reproduces the complete mass defect. The relevant degrees of freedom are the massless $(4,4)$ strings. Both results are in contrast with existing string field theory calculations. The excitation spectrum of the small instanton is found to be equal to the excitation spectrum of the fluxon solution on $\mathbb{R}^2_\theta \times \mathbb{R}$ which we trace back to T-duality. For the effective theory of the $(0,0)$ string excitations we obtain a BFSS matrix model. The number of states in the instanton background changes significantly when the B-field becomes self-dual. This leads us to the proposal of the existence of a phase transition or cross over at self-dual B-field.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Tachyon Condensation, Non-Commutative Geometry.

Contents

1. Introduction

The striking correspondence between superstring theories with D-branes and supersymmetric gauge theories led to a number of insights in the structure of supersymmetric gauge theories. This correspondence becomes even more interesting if a constant B-field background is taken into account in string theory. The resulting low energy description is a noncommutative gauge theory where the noncommutative structure of space is determined by the B-field background [1]. In addition to α', the B-field introduces a new scale into the theory and in some cases much more of the structure of the string theory survives the low energy limit, where $\alpha' \to 0$. This is observed as a large number of nontrivial classical solutions in the gauge theory and also the excitation spectrum which survives this limit shows much more diversity than in the case without a B-field background. Hence there is some legitimate hope that from noncommutative field theory one learns something about string theory, using the somewhat simpler techniques of (noncommutative) quantum field theory. In the generic case the resulting noncommutativity also acts as a regulator at small distances. For example the small instanton singularity in the moduli space of instantons is resolved for a generic B-field background.

In this paper we consider supersymmetric U(1) Yang-Mills theory defined on $\mathbb{R}^4_\theta \times \mathbb{R}$ or $\mathbb{R}^4_\theta$. We investigate various (quantum) aspects of nontrivial solutions on $\mathbb{R}^4_\theta$, which are solitonic states in the five dimensional case or (unstable) "instantons" in the latter case. The D-brane setup for such configurations is a $D0$-$D4$-brane system where the rank four B-field has nontrivial components along the $D4$-brane.

The different aspects that we address for this system can be summarised in a cycle of different states of this system. Initially, for generic B-field the $D0$-$D4$ system is unstable and described by a solitonic unstable solution of the Yang-Mills equations. We review the spectrum of this system taking nontrivial scalar field backgrounds into account. They are identified with the distance of the $D0$-brane from the $D4$-brane in the transverse directions. If the separation of the two branes becomes smaller than the size of the solitonic state the system becomes unstable and a tachyonic mode appears. Then the system condensates to a BPS $D0$-$D4$ bound state. We calculate the tachyon potential and identify the massless $(4,4)$ strings as the relevant degrees of freedom in the tachyon condensation process. The depth of the tachyon potential equals exactly the mass defect in forming the $D0$-$D4$ bound state in the considered Seiberg-Witten limit. Both observations for the tachyon condensation are in contrast to string field theory computations [2] as we will discuss. The BPS $D0$-$D4$ bound state is described by a noncommutative instanton with a non-self-dual B-field background. We compute the exact excitation spectrum for this state and show that it is quantitatively and qualitatively very different from the spectrum of the unstable state. In the next step we tune the B-field to become self-dual. The $D0$-$D4$ bound state instanton solution smoothly approaches the small instanton which is described by a regular self-dual solution although the moduli space has a small instanton singularity as in the commutative case. We discuss the various zero-mode moduli of the small instanton solution and its excitation spectrum. Although the classical solutions are continuously connected by this B-field limit the spectra are not. The number of states is significantly different. New scalar field zero-modes show that the $D0$-brane deconfines and can separate from the $D4$-brane. We also find that the effective theory for $D0$-$D0$ string excitations is described by a BFSS matrix model. We observe that the spectrum of the small instanton coincides with the spectrum of the fluxon solution on $\mathbb{R}^2_\theta \times \mathbb{R}$ [3, 4] and argue, based also on the ADHM construction, that this is due to T-duality. Because of the significant change in the number of states we propose the existence of a phase transition or crossover in the noncommutative super Yang-Mills theory when the B-field becomes self-dual and the $D0$-brane separates from the $D4$-brane. In principle one can now reinitiate this circuit by turning on again an anti-self-dual part of the B-field and driving the system into its unstable state.

For technical reasons the paper is not organised in the circuit as described above but as follows: In section 2 we review some basic facts about noncommutative field theory to set up our notation. In section 3 we show how our formalism includes lower dimensional

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solitons. In section 4 we discuss the unstable state and the small instanton. In section 5 we investigate the instanton solution for non-self-dual B-field. In section 6 we discuss the tachyon condensation and compute the tachyon potential. In section 7 we draw our conclusions and present a proposal as well as a puzzle.

2. Noncommutative field theory

To set up our notation we briefly review some of the basic results of noncommutative field theory. Of particular interest are noncommutative Yang-Mills theories which arise as a certain zero slope limit $\alpha' \to 0$ of open string theory in a B-field background [1].

In this so called Seiberg-Witten limit the three point amplitudes of mass-less open strings on a stack of N Dp-branes are reproduced by the noncommutative U(N) Yang-Mills action

$$\frac{1}{2g_{YM}^2}\int dx^{p+1} G^{MP} G^{NQ}\,\mathrm{tr}\{F_{MN}\star F_{PQ}\} + \mathcal{O}(\alpha')\,, \tag{2.1}$$

where the open string metric G_{MN} in the Seiberg-Witten limit is expressed through the closed string metric g_{MN} and the B-field B_{MN} in the following way:

$$G_{MN} \to -(2\pi\alpha')^2 (Bg^{-1}B)_{MN}\,. \tag{2.2}$$

We take the gauge fields to be anti-hermitian and the anti-hermitian generators of the U(N) gauge group are normalised to $\mathrm{tr}\{T^aT^b\} = -\frac{1}{2}\delta^{ab}$. The field strength

$$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]_\star \tag{2.3}$$

is also defined w.r.t. the star-product "$\star$", which for constant θ^{MN} is given by

$$(f\star g)(x) := e^{\frac{i}{2}\theta^{MN}\partial_M\partial'_N} f(x)g(x')|_{x'=x}\,. \tag{2.4}$$

In the Seiberg-Witten limit the noncommutativity parameter and the Yang-Mills coupling are related to the closed string coupling g_s and the background B-field as follows:

$$\theta^{MN} \to (B^{-1})^{MN}\,, \qquad \frac{1}{g_{YM}^2} \to \frac{\alpha'^{\frac{3-p}{2}}}{(2\pi)^{p-2} g_s\sqrt{\det(2\pi\alpha' Bg^{-1})}}\,. \tag{2.5}$$

In matching instanton quantities with D-brane quantities we will need these relations.

Noncommutative geometry is therefore defined through the algebra $\mathcal{A}_\theta$ of functions on a manifold with the multiplication given by a star-product. Our case of interest are $D4$-branes with a non-vanishing B-field along the four (spatial) dimensions of the branes. This corresponds to the noncommutative $\mathbb{R}^4_\theta$, which we parametrise by cartesian coordinates $x^{\mu=1,\dots,4}$. The star-product (2.4) leads to the fundamental relation for the algebra $\mathcal{A}_\theta$,

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}\,. \tag{2.6}$$

In [1] the string spectrum for the $D0$-$D4$ brane system was computed for a flat target space, i.e. with the closed string metric $g_{\mu\nu} = \delta_{\mu\nu}$. We will rather work in cartesian coordinates with respect to the open string metric (2.2), i.e. $G_{\mu\nu} = \delta_{\mu\nu}$. This involves a B-field dependent coordinate transformation (see below).

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Operator formalism. The algebra $\mathcal{A}_\theta$ can also be represented in a different way. When representing the generators x^μ by operators $\hat{x}^\mu$ acting on a Hilbert space $\mathcal{H}$ the Weyl group associated to the Lie algebra (2.6) consists of elements

$$U_p := e^{ip_\mu \hat{x}^\mu} \qquad \text{with } \operatorname{Tr}_{\mathcal{H}}\{U_p U_{p'}^\dagger\} = \frac{(2\pi)^{d/2}}{\operatorname{Pf}(\theta)}\, \delta^d(p-p')\,, \tag{2.7}$$

where the trace is taken over the Hilbert space $\mathcal{H}$ and Pf denotes the pfaffian. The dimension $d = 4$ in our case. The operator U_p generates translations on the algebra $\mathcal{A}_\theta$, i.e.

$$[U_p, \hat{x}^\mu] = \theta^{\mu\nu} p_\nu\, U_p\,. \tag{2.8}$$

Ordinary commutative functions are mapped to operators by the Weyl map:

$$f(x) \longmapsto \hat{f}(\hat{x}) = \int \frac{d^dx\; d^dp}{(2\pi)^d}\; f(x)\; e^{-ipx} U_p\,. \tag{2.9}$$

Under the Weyl map (2.9) the star-product of commutative functions becomes the ordinary product of operators. Differentiation and integration become the following operations on operators acting on $\mathcal{H}$:

$$\partial_\mu \;\longmapsto\; -i(\theta^{-1})_{\mu\nu}[\hat{x}^\nu,\ .\,]\,, \qquad \int d^dx \;\longmapsto\; (2\pi)^{d/2}\ \operatorname{Pf}(\theta)\operatorname{Tr}_{\mathcal{H}}\,. \tag{2.10}$$

Fock space. Let us now focus on $\mathbb{R}^4_\theta$. The matrix $\theta^{\mu\nu}$ in (2.6) is antisymmetric. Thus by an orthogonal transformation it can be brought to the canonical form

$$(\theta^{\mu\nu}) = \begin{pmatrix} & -\theta_1 & & \\ \theta_1 & & & \\ & & & -\theta_2 \\ & & \theta_2 & \end{pmatrix}, \tag{2.11}$$

where due to of the above mentioned transformation to cartesian coordinates w.r.t. the open string metric (2.2) the noncommutativity parameters are related to the B-field components used in section 5 of [1] as follows:

$$\theta_i = 2\pi\alpha' b_i\,. \tag{2.12}$$

In the following we assume that $\theta_{1,2} > 0$. We will discuss the regime of these parameters in more detail below. The form of $\theta^{\mu\nu}$ suggests a natural decomposition of the noncommutative space into noncommutative planes reflecting the residual isometries in the presence of a B-field background. On each plane we introduce complex coordinates,[1] $z_1 = x^1 - ix^2$, $z_2 = x^3 - ix^4$ such that the gauge fields write as

$$A_{z_1} = \frac{1}{2}(A_1 + iA_2)\,, \qquad \bar{A}_{\bar{z}_1} = \frac{1}{2}(A_1 - iA_2) = -(A_{z_1})^\dagger\,, \tag{2.13}$$

and analogously for the second plane parametrised by z_2. After introducing the the operators

$$c_\alpha := \frac{1}{\sqrt{2\theta_\alpha}}\; z_\alpha\,, \qquad c_\alpha^\dagger := \frac{1}{\sqrt{2\theta_\alpha}}\; \bar{z}_\alpha\,, \tag{2.14}$$

[1] In the following we omit the indication of operators by hats. We follow the notation of [3].

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the algebra (2.6) becomes a two-oscillator algebra and one can build up the Fock space as follows:

$$[c_\alpha, c_\beta^\dagger] = \delta_{\alpha\beta}\,, \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\,, \qquad \mathcal{H}_\alpha = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle_\alpha\,, \tag{2.15}$$

where $|n\rangle_\alpha$ are the usual occupation number states.

With the identification (2.10) the covariant derivatives in complex coordinates are the represented by the operators

$$\nabla_{z_\alpha} = -\frac{1}{\sqrt{\xi_\alpha}}\, c_\alpha^\dagger + A_{z_\alpha}\,, \qquad \bar{\nabla}_{\bar{z}_\alpha} = \frac{1}{\sqrt{\xi_\alpha}}\, c_\alpha + \bar{A}_{\bar{z}_\alpha} = -(\nabla_{z_\alpha})^\dagger\,, \tag{2.16}$$

where we have introduced the abbreviation $2\theta_\alpha =: \xi_\alpha$. Because of noncommutativity and (2.10) the covariant derivatives for all fields in the adjoint representation act as $\nabla_{z_\alpha}\Phi := [\nabla_{z_\alpha}, \Phi]$, even if the gauge group is taken to be U(1). The field strength in terms of the derivatives (2.16) writes as

$$\begin{aligned} F_{z_\alpha,\bar{z}_\alpha} &= [\nabla_{z_\alpha}, \bar{\nabla}_{\bar{z}_\alpha}] - \frac{1}{\xi_\alpha} \\ F_{z_1,z_2} &= [\nabla_{z_1}, \nabla_{z_2}]\,, \qquad F_{z_1,\bar{z}_2} = [\nabla_{z_1}, \bar{\nabla}_{\bar{z}_2}]\,. \end{aligned} \tag{2.17}$$

3. A self-dual descent

In the following we consider a gauge field

$$A_M = (A_0, A_\mu, X_m)\,, \qquad \mu = (1, 2, 3, 4)\,, \tag{3.1}$$

in ten or six dimensions, respectively. This depends on the supersymmetry content which one considers in four dimensions, obtained after dimensional reduction. As is well known [5] the dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills theory from ten/six to four dimensions leads to $\mathcal{N} = 4/\mathcal{N} = 2$ super Yang-Mills theory. We describe two objects simultaneously. The five brane, which is an instanton solution embedded as a soliton in the 4+1 dimensional theory contained in ten-dimensional super Yang-Mills theory. It carries a five-form charge [6] and corresponds to a $D0$-$D4$ brane system. The second object is an instanton solution of euclidean four-dimensional Yang-Mills theory also embedded in (possibly euclidean) ten-dimensional super Yang-Mills theory. The instanton solution of euclidean four-dimensional Yang-Mills theory corresponds to a $D(-1)$-$D3$ brane system. In this case the quantum theory is well defined. This is possibly also true on noncommutative space, since it seems that the θ-deformation does not spoil the renormalisation of divergences due to planar loop graphs [7−9]. For supersymmetric Yang-Mills theories also the infrared divergences (UV/IR mixing) due to non-planar loop contributions are under much better control, see e.g. [10, 11].

To get down to the physically relevant $3 + 1$ dimensions in the case of the five brane one would also have to compactify one of the nontrivial directions. Thus the five brane is more correctly described by a caloron than an instanton. But we will not discuss this case here. Anyway, the two systems, $D0$-$D4$ and $D(-1)$-$D3$, are formally very similar and we will usually use the notation for the five brane but use the term instanton. To obtain the euclidean four-dimensional case one just has to set the temporal component of the gauge field, classical and quantum, as well as time derivatives identically to zero in the following sections. For classical solutions this just corresponds to static solutions in the temporal gauge $A_0 = 0$.

We briefly review further dimensional reductions to show how lower dimensional objects like monopoles/fluxons and vortices are contained as special cases in the following relations. For the rest of the paper we indicate classical configurations with caligraphical letters. Covariant derivatives w.r.t. classical background gauge fields are denoted by D_μ instead of ∇_μ.

The instanton on $\mathbb{R}^4_\theta$. In this case the non-vanishing components of the gauge field $\mathcal{A}_M$ are $\mathcal{A}_\mu$ which depend only on the noncommutative coordinates x^μ. For the five brane this corresponds to static solutions in temporal gauge. The noncommutative instanton is then described by the four-dimensional self-duality equations[2] $*\mathcal{F} = \mathcal{F}$. In cartesian coordinates they are

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathcal{F}^{\alpha\beta}\,. \tag{3.2}$$

The monopole on $\mathbb{R}^2_\theta \times \mathbb{R}$. Now we write the nontrivial gauge field as $\mathcal{A}_\mu = (\mathcal{A}_{i=1\ldots3}, \phi)$ and let the coordinates $x^{3,4}$ be commutative. Then the dimensional reduction w.r.t. to the x^4-direction in the self-duality equations (3.2) leads to:

$$\mathcal{F}_{ij} = \varepsilon_{ijk} D_k \phi\,. \tag{3.3}$$

These are the equations for the static monopole, again in temporal gauge.

The vortex on $\mathbb{R}^2_\theta$. Reducing further the Bogomolnyi equations (3.3) with respect to the commutative coordinate x^3 one obtains the vortex equation on the noncommutative plane. Writing the nontrivial gauge field components as $\mathcal{A}_\mu = (\mathcal{A}_{a=1,2}, \phi_1, \phi_2)$ and introducing the complex field $\phi = \frac{1}{2}(\phi_1 + i\phi_2)$ one obtains from (3.3):

$$\mathcal{F}_{ab} = 2i\ \varepsilon_{ab}[\phi, \bar{\phi}]\,, \qquad D_-\phi = 0\,, \tag{3.4}$$

where bar means complex conjugation and $D_- = \frac{1}{2}(D_1 + iD_2)$. These are (2+1)-dimensional static vortex equations in the temporal gauge. They differ from the noncommutative ANO-vortex equations which include a Fayet-Iliopoulos term [12].

In the following we will exclusively treat the instanton on $\mathbb{R}^4_\theta$ but a number of the formal manipulations will be valid also for the lower-dimensional objects through the identification we gave above.

[2] Nowadays the term instanton is usually used for anti-self-dual solutions. For linguistic simplicity and to avoid some signs we consider self-dual solutions. From the viewpoint of the geometry of $K3$ Calabi-Yau manifolds anti-self-dual fields appear to be more convenient. But the results that we obtain are mutatis mutandis valid for anti-self-dual backgrounds. Correctly, self-dual solutions are identified with anti-$D0$-$D4$ branes but we keep our "inexact" notion.

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3.1 The small instanton singularity regime

In what follows we restrict ourself to a single $D4$-brane, i.e. the gauge group is U(1). Thus the nontrivial solutions (instantons) that we will consider have no counterpart in the commutative theory. The physical properties of these solutions and the possible quantum states will crucially depend on the properties of the noncommutative parameters $\theta^{\mu\nu}$. The most important characteristic is the sign of the pfaffian $\mathrm{Pf}(\theta) = \theta_1\theta_2$. In the case that $\mathrm{Pf}(\theta) > 0$ $\theta^{\mu\nu}$ is continuously connected to the self-dual point $\theta^{\mu\nu}_{SD}$, where $\theta_1 = \theta_2$. Therefore we call this the self-dual regime of $\theta^{\mu\nu}$. For $\mathrm{Pf}(\theta) < 0$ the situation is reversed and $\theta^{\mu\nu}$ is continuously connected to the anti-self-dual point $\theta^{\mu\nu}_{ASD}$, where $\theta_1 = -\theta_2$. In the past several authors constructed U(1) instanton solutions on $\mathbb{R}^4_\theta$ with opposite duality for the field strength $\mathcal{F}_{\mu\nu}$ and the noncommutativity parameter $\theta^{\mu\nu}$ [13–16]. For the rest of the paper we will concentrate on the opposite case. We assume $\theta^{\mu\nu}$ to be in the self-dual regime, i.e. $\mathrm{Pf}(\theta) > 0$, where we approach the special point $\theta^{\mu\nu}_{SD}$ from values $\theta_1 > \theta_2$. The field strength $\mathcal{F}_{\mu\nu}$ will be either self-dual for general $\theta^{\mu\nu}$ in the self-dual regime or become self-dual in the special point $\theta^{\mu\nu}_{SD}$. The latter case corresponds to the small instanton singularity.

4. Unstable solitonic states

Decomposing the gauge field according to (3.1) the dimensionally reduced (super) Yang-Mills action (2.1) writes as

$$S = \frac{1}{4g^2_{YM}} \int \mathrm{Tr}\Big\{(F_{\mu\nu})^2 + 2(\nabla_\mu X_m)^2 + [X_m, X_n]^2 - \\ - 2(\nabla_0 X_m)^2 - 2(\nabla_\mu A_0 - \partial_t A_\mu)^2\Big\} + \text{fermions}\,, \tag{4.1}$$

where we have introduced the abbreviation

$$\int \mathrm{Tr} := \int dt\, (2\pi)^2 \mathrm{Pf}(\theta)\, \mathrm{Tr}_{\mathcal{H}}\ . \tag{4.2}$$

When restricting to the (euclidean) four-dimensional case one has to omit the integration over time in (4.2). For later convenience we write down the fermionic part of the action using the higher dimensional notation:

$$S_{\text{ferm}} = -\frac{1}{2g^2_{YM}} \int \mathrm{Tr}\{\bar\lambda \Gamma^M \nabla_M \lambda\}\,, \tag{4.3}$$

where $\bar\lambda = \lambda^\dagger \Gamma^0$. For most of our considerations we will assume $\mathcal{N} = 4$ supersymmetry[3] so that λ is a Majorana-Weyl spinor in ten-dimensional space-time. In the case of $\mathcal{N} = 2$ supersymmetry λ is a Weyl spinor in six-dimensional space-time. The action is invariant under the same susy-transformations as in the commutative case:[4]

$$\delta A_M = \frac{1}{2}\left(\bar\lambda \Gamma_M \epsilon_1 - \bar\epsilon_2 \Gamma_M \lambda\right)$$

[3]Here the specific name for the extended supersymmetries refers to the four-dimensional case.

[4]Note that the susy currents cannot be determined by local susy transformations. In this case the susy parameters became nontrivial operators and relevant manipulations are no longer possible.

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$$\delta\lambda = \frac{1}{2} F_{MN}\Gamma^{MN}\epsilon_1\,, \qquad \delta\bar{\lambda} = -\frac{1}{2}\bar{\epsilon}_2\Gamma^{MN}F_{MN}\,, \tag{4.4}$$

where $\Gamma^{MN} = \frac{1}{2}[\Gamma^M, \Gamma^N]$ and the supersymmetry parameters are proportional to the unit operator. For $\mathcal{N} = 4$ supersymmetry $\epsilon_1 = \epsilon_2$. In the following we will mostly concentrate on the bosonic fields. The fermionic fields are either determined by supersymmetry or put to zero if not stated differently. In the operator formalism the noncommutative U(1) gauge symmetry of the action is represented by unitary transformations $U_{\mathcal{H}}$ acting on the Hilbert space (2.15):

$$A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U\,, \qquad X_m \to U^{-1}X_m U\,, \tag{4.5}$$

with $U^{-1}U = UU^{-1} = \mathbb{1}$. The operator ∂_μ is given in (2.10).

The equations of motion for static configurations in the temporal gauge are

$$\begin{aligned} D^\mu \mathcal{F}_{\mu\nu} + [D_\nu \mathcal{X}_m, \mathcal{X}_m] &= 0\,, \\ \mathscr{D}^2 \mathcal{X}_m + [[\mathcal{X}_m, \mathcal{X}_n], \mathcal{X}_n] &= 0\,. \end{aligned} \tag{4.6}$$

Here and in the following $\mathscr{D}^2 = D^\mu D_\mu$. Self-dual gauge fields, for which $(*\mathcal{F})_{\mu\nu} := \tilde{\mathcal{F}}_{\mu\nu} = \mathcal{F}_{\mu\nu}$, with non-vanishing scalar fields $\mathcal{X}_m$ are solutions to these equations if the scalar fields $\mathcal{X}_m$ are *commuting zero-modes* of the operator D_μ, i.e.

$$D_\mu \mathcal{X}_m = 0\,, \qquad [[\mathcal{X}_m, \mathcal{X}_n], \mathcal{X}_n] = 0\,. \tag{4.7}$$

If $[\mathcal{X}_m, \mathcal{X}_n] = 0$ these commuting zero-modes do not contribute to the action (4.1). The (four-dimensional) classical action for such solutions is given by

$$S^{cl} = \frac{1}{4g_{YM}^2}\int \mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\,, \tag{4.8}$$

and therefore the commuting zero-modes (4.7) are moduli of self-dual solutions. We will discuss this in more detail below.

Generating solutions. The so called solution generating technique generates nontrivial solutions from vacuum configurations [17]. This method uses non-unitary isometries of the equations of motion to generate new configurations which are not gauge-equivalent to the vacuum. An important input in this context are the so called shift operators. For the noncommutative $\mathbb{R}^4_\theta$ a shift operator of order k can be defined as a product of order one shift operators

$$\begin{aligned} S^{\mathrm{I}} &= \sum_{m=0,n=1} |m,n\rangle\langle m,n| + \sum_{m=0} |m+1,0\rangle\langle m,0|\,, \\ S^{\mathrm{II}} &= \sum_{m=1,n=0} |m,n\rangle\langle m,n| + \sum_{n=0} |0,n+1\rangle\langle 0,n|\,. \end{aligned} \tag{4.9}$$

These operator shift states $|m,n\rangle \in \mathcal{H}$ in the first and second occupation number, respectively. Different sequences of k factors of $S^{\mathrm{I,II}}$ define different shift operators of order k which are nevertheless gauge equivalent [18]. The relevant properties of an order k shift operator are

$$S_k^\dagger S_k = \mathbb{1}\,, \qquad S_k S_k^\dagger = \mathbb{1} - P_k\,, \qquad P_k S_k = 0\,, \tag{4.10}$$

where P_k is a projection operator being the unit on the subspace $V_k = P_k\mathcal{H} \subset \mathcal{H}$ on which S_k acts non-trivially, so that[5]

$$\operatorname{Tr} P_k = k\,. \tag{4.11}$$

With this preparation one can immediately write down charge k solutions generated from the vacuum $D_\mu^{\rm vac} = -i(\theta^{-1})_{\mu\nu}x^\nu$ and the associated field strength [19]:[6]

$$D_\mu = iS_k(\theta^{-1})_{\mu\nu}x^\nu S_k^\dagger\,, \qquad \mathcal{F}_{\mu\nu} = -i(\theta^{-1})_{\mu\nu}\,P_k\,. \tag{4.12}$$

For self-dual $\theta_{\mu\nu}$ the field strength (4.12) is self-dual. Expressed in complex coordinates this solution writes as

$$D_\alpha = \frac{1}{\sqrt{\xi_\alpha}}\;S_k c_\alpha^\dagger S_k^\dagger\,, \quad \bar{D}_\alpha = -\frac{1}{\sqrt{\xi_\alpha}}\;S_k c_\alpha S_k^\dagger\,, \tag{4.13}$$

where $\alpha = 1,2$ refers to $z_{1,2}$. The classical action (4.8) or mass and the instanton charge for this solution are easily obtained as (here the time-integral is not included in $\int \operatorname{Tr}$)

$$\begin{aligned} M_5 &= \frac{4\pi^2\,k}{2g_{YM}^2}\left(\frac{\theta_1}{\theta_2}+\frac{\theta_2}{\theta_1}\right), \\ Q &= -\frac{1}{16\pi^2}\int \operatorname{Tr} F_{\mu\nu}(*F)^{\mu\nu} = k\,, \end{aligned} \tag{4.14}$$

where M_5 denotes the classical mass of the five brane. In fact the notion of mass is somewhat misleading in this case since, as we will see later, this configuration is not stable for generic $\theta_{\mu\nu}$. So it would be more appropriate to use the term energy. Note that at the self-dual point $\theta_1 = \theta_2$ the classical five-brane mass and thus the classical noncommutative U(1) instanton action equals half of the k-instanton action $S_{\rm inst} = \frac{8\pi^2}{g_{YM}^2}k$ for (commutative) U(2) instantons [21].

D-brane interpretation. In [19] the D-brane interpretation of the solutions (4.12) was conjectured. This is based on the matching of the (classical) binding energies for the five brane and the (anti) $D0$-$D4$ brane system. In the noncommutative Yang-Mills description the binding energy for a single $D0$-brane is given by

$$E_{\rm bind} = M_5 - M_5^{BPS} = \frac{2\pi^2}{g_{YM}^2}\left(\sqrt{\frac{\theta_1}{\theta_2}}-\sqrt{\frac{\theta_2}{\theta_1}}\right)^2 = \frac{1}{2g_s\sqrt{\alpha'}}\left(\frac{1}{b_1}-\frac{1}{b_2}\right)^2, \tag{4.15}$$

where in the second equation we have written the binding energy in terms of the string quantities (2.12) and (2.5) for $p = 4$. In units where $\alpha' = 1$ the mass defect of the formation of a $D0$-$D4$ bound state in the limit $b_i \gg 1$ is given by [2][7]

$$\Delta M = M_{D0} + M_{D4} - M_{D0+D4} = \frac{1}{2g_s}\left(\frac{1}{b_1}-\frac{1}{b_2}\right)^2, \tag{4.16}$$

and thus matches exactly the binding energy (4.15). We will see below that also the excitation spectra of the solitonic solution (4.12) match the one of the $D0$-$D4$ system.

[5] Drawing the states $|m,n\rangle$, $m,n = 0,1,2\ldots$, on a plane with lattice points (m,n), $\operatorname{Tr} P_k$ is equal to the "area" of the subspace V_k in this picture.

[6] For similar constructions of U(1) instantons in the signature $2+2$ see [20].

[7] Our notation is related to the one of [2] as follows: $b_1 = b$, $b_2 = -b'$ and $g_s = 2\pi^2 g$. The range $b_i \gg 1$ of the B-field corresponds to "case (3)" in [2].

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4.1 Zero-mode moduli

From the properties (4.10) one immediately sees that the classical equations of motion (4.6) are fulfilled by (4.12) and the projector P_k projects on the space of zero-modes given by

$$D_\mu \mathcal{X} = 0 \qquad \text{for } \mathcal{X} \in V_k \otimes V_k^* \,. \tag{4.17}$$

Up to gauge-equivalence the space V_k is spanned by the states $|j\rangle := |m,n\rangle$ with

$$j = m + \frac{(m+n)(m+n+1)}{2}\,, \qquad j = 0, \dots, k-1\,. \tag{4.18}$$

So V_k is the lower left corner of the $(m, n \geq 0)$-plane with the "area" $\mathrm{Tr}(P_k) = k$. Thus a general zero-mode is of the form

$$\mathcal{X} = \sum_{j,j'=0}^{k-1} \mathcal{X}_{jj'} |j\rangle\langle j'|\,. \tag{4.19}$$

The solution (4.12) breaks the $\mathrm{U}(\mathcal{H})$ gauge symmetry down to $\mathrm{U}(k)$ which acts non-trivially only in the subspace $V_k \otimes V_k^*$:

$$\mathrm{U}(k) \ni U = \sum_{j,j'=0}^{k-1} U_{jj'} |j\rangle\langle j'|\,. \tag{4.20}$$

Stated differently, these are the transformations which leave the gauge connections (4.12) invariant and are therefore the analogue of the stability group with corresponding gauge orientation zero-modes for commutative $\mathrm{U}(2)$ instantons [21].

The zero-modes (4.17) transform under the residual gauge symmetry in the adjoint representation:

$$\mathcal{X} \to U^\dagger \mathcal{X} U = \sum_{i,j=0}^{k-1} (U_{i'i})^\dagger \mathcal{X}_{i'j'} U_{j'j} |i\rangle\langle j| \tag{4.21}$$

where summation over repeated matrix indices is implied. Of special interest are *commuting* zero-modes (4.7) which are possible nontrivial scalar field solutions. The simplest solution to (4.7) is given by diagonal matrices in the representation (4.19):

$$(\mathcal{X}^m)_{jj'} = \mathrm{diag}\Big(\alpha_0^{(m)}, \dots, \alpha_{k-1}^{(m)}\Big)\,. \tag{4.22}$$

We will see later that the vectors α_j^m with components in the m directions transverse to the $D4$-brane can be interpreted as the positions of the k $D0$-branes outside the $D4$-brane. We also see that if the k $D0$-branes are separated, i.e. $\alpha_0^{(m)} \neq \cdots \neq \alpha_{k-1}^{(m)}$, the residual $\mathrm{U}(k)$ gauge symmetry is further broken down to $\mathrm{U}(1)^k$.

In general the (second) equation in (4.7) allows more nontrivial solutions for $k \times k$ matrices. In fact these are the equations for static bosonic configurations of the BFSS matrix quantum mechanics which has been conjectured to be equivalent to M-theory [22]. Nontrivial solutions to (4.7) may for example describe $D0$-$D4$ bound states without a B-field background [23]. We will give a more detailed description of the relation to the matrix model below.

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4.2 Fluctuation spectrum

In this section we investigate the fluctuation spectrum or equivalently the asymptotic states of the quantum field theory in the presence of the above discussed background solutions. For this we expand the full fields around the classical background (4.12):

$$A_\mu \to \mathcal{A}_\mu + a_\mu\,, \qquad A_0 \to a_0\,, \qquad X_m \to \mathcal{X}_m + X_m\,. \tag{4.23}$$

In addition we need a gauge fixing condition. As has been shown in the quantisation of commutative monopoles and vortices the most comfortable gauge in the presence of classical backgrounds is the covariant background gauge implemented in the higher dimensional space [24, 25]:

$$\begin{aligned} S_{\rm gf} &= \frac{1}{2g_{YM}^2}\int \mathrm{Tr}(D^M a_M)^2 \\ &= \frac{1}{2g_{YM}^2}\int \mathrm{Tr}(-\dot{a}_0 + D_\mu a_\mu + [\mathcal{X}_m, X_m])^2\,. \end{aligned} \tag{4.24}$$

By the usual procedure which can be applied without any problems to the noncommutative case one obtains the ghost action associated to (4.24):

$$S_{\rm FP} = -\frac{1}{g_{YM}^2}\int \mathrm{Tr}(bD^M D_M c + bD^M[a_M, c])\,. \tag{4.25}$$

We give here the ghost action for future reference, since it will be needed for one-loop corrections.

We focus on the $k = 1$ background, such that

$$P_{k=1} =: P = |00\rangle\langle 00|\,. \tag{4.26}$$

Without loss of generality we can choose the $D0$-brane to be separated in the $m = 1$ direction from the $D4$-brane. Thus the only nontrivial scalar field is

$$\mathcal{X}_1 = i\mu P\,, \tag{4.27}$$

where $\mu \in \mathbb{R}$ since all fields are anti-hermitian. Including the gauge-fixing action (4.24) the action quadratic in the fluctuations reads as

$$\begin{aligned} S^{(2)} =& \frac{1}{2g_{YM}^2}\int \mathrm{Tr}\, a_\mu \left\{\delta^{\mu\nu}(\partial_t^2 - \mathscr{D}^2 + \mu^2[P,[P,.\,]]) - 2[\mathcal{F}^{\mu\nu},.\,]\right\} a_\nu + \\ &+\frac{1}{2g_{YM}^2}\int \mathrm{Tr}\, X_m \left\{(\partial_t^2 - \mathscr{D}^2) + \mu^2[P[P,.\,]]\right\} X_m - \\ &-\frac{1}{2g_{YM}^2}\int \mathrm{Tr}\, a_0 \left\{(\partial_t^2 - \mathscr{D}^2) + \mu^2[P[P,.\,]]\right\} a_0 + \text{ghosts} + \text{fermions}. \end{aligned} \tag{4.28}$$

In the following we will mostly concentrate on the bosonic fluctuations. The fermionic ones can be determined by supersymmetry, at least at the self-dual point $\theta_1 = \theta_2$. The fluctuation operators for the different fields are very similar, except for the additional spin-coupling through the background field $\mathcal{F}^{\mu\nu}$ for the "gluons" a_μ. The scalar field background (4.27), i.e. the separation of the $D0$-brane, induces a mass term for all quantum fields. The a_0 fluctuations are not physical and will be cancelled by the ghost. But let us first look at the scalar fields.

Scalar fluctuations. We look for the spectrum (discrete and continuous) of the fields X_m and a_0. Thus we have to solve the eigenvalue problem

$$-\Delta\varphi_\omega = \omega^2\varphi_\omega \qquad \text{with } -\Delta = -\mathscr{D}^2 + \mu^2[P,[P,\ .\ ]]\,. \tag{4.29}$$

The string picture as well as the form of the background (4.12), (4.13) suggest the separation of the eigenmodes φ_ω into different sectors of the Hilbert space $V = \mathcal{H}\otimes\mathcal{H}^*$:

$$\varphi_\omega := \alpha P + PaS^\dagger + SbP + SdS^\dagger\,, \tag{4.30}$$

where $\alpha \in \mathbb{C}$ and a, b, d are general operators $\in V$. In the following we choose the shift operator $S = S^{\mathrm{II}}$ in (4.9). Equation (4.30) corresponds to a decomposition $V = V_{00}\oplus V_{04}\oplus V_{40}\oplus V_{44}$ according to open strings on the $D0$-brane, stretched between the $D0$-$D4$-brane and open strings living on the $D4$-brane. This is quite analogous to the noncommutative $\mathbb{R}^2_\theta$ as it was considered in [19, 3]. But as we will see, the slightly different definition of the sectors in (4.30) will simplify the analysis especially for more involved situations which we will consider below as well as for higher dimensional noncommutative spaces. Compare also the appendix of [26]. Recalling the property (4.10) one can see that V_{00} and V_{44} do not feel the mass term in (4.29). With the decomposition (4.30) one immediately obtains the spectrum of the V_{00} and the $V_{04,40}$ sectors. The eigenvalue equation (4.29) for the V_{44} sector in the background (4.13) becomes

$$2\sum_{\alpha=1,2}\frac{1}{\xi_\alpha}\Big([c_\alpha,[c_\alpha^\dagger,d\,]] + [c_\alpha^\dagger,[c_\alpha,d\,]]\Big) = \omega^2 d\,. \tag{4.31}$$

As expected this is just the free laplacian since the background (4.13) is related by an isometry to the vacuum. Using the fact that U_p generates translations (2.8) and introducing the complex momenta $k_{\alpha=1,2} := \frac{1}{2}(k_{1,3} + i\ k_{2,4})$ one can easily see that the anisotropic noncommutative plane wave (2.7)

$$d = U_k = e^{i\sum_\alpha \sqrt{\xi_\alpha}(k_\alpha c_\alpha + \bar{k}_\alpha c_\alpha^\dagger)}\,, \tag{4.32}$$

is a solution to (4.31) with eigenvalue $\omega^2 = k_\mu k^\mu$. Collecting things together we have the following spectrum of scalar fluctuations $(m, n \geq 0)$

$$\begin{aligned}
V_{00}: \omega^2 &= 0\,, & \varphi_0 &= P\,,\\
V_{04}: \omega^2 &= \frac{2m+1}{\theta_1} + \frac{2n+1}{\theta_2} + \mu^2\,, & \varphi_{m,n} &= |0,0\rangle\langle mn|S^\dagger\,,\\
V_{44}: \omega^2 &= k_\mu k^\mu\,, & \varphi_k &= \tfrac{\sqrt{\theta_1\theta_2}}{2\pi}\,SU_kS^\dagger\,.
\end{aligned} \tag{4.33}$$

The V_{40} sector is simply given by the hermitian conjugate of the V_{04} states. The fluctuations are ortho-normalised w.r.t. the norm $\mathrm{Tr}(\varphi_\omega^\dagger\ \varphi_{\omega'})$.

Gauge field fluctuations. The gauge fields a_μ are now governed by the following equation:

$$-\Delta_{\mu\nu}\ a_\nu = \omega^2\ a_\mu \qquad \text{with } -\Delta_{\mu\nu} = -\delta_{\mu\nu}\Delta + 2i\theta^{-1}_{\mu\nu}[P,.\,]\,, \tag{4.34}$$

where Δ was already defined in (4.29) and we have used (4.12). The additional spin coupling acts only in the $V_{04,40}$ sector. Thus, up to polarisation vectors ε_μ, the gauge fields a_μ of the V_{00} and V_{44} sector are given by the states and eigenvalues (4.33). With the canonical form (2.11) of $\theta^{\mu\nu}$ the operator $\Delta_{\mu\nu}$ becomes block-diagonal with the two by two blocks

$$-\tilde{\Delta}_{\alpha=1,2} = \begin{pmatrix} -\Delta & \frac{i}{\xi_\alpha}[P, .\,] \\ -\frac{i}{\xi_\alpha}[P, .\,] & -\Delta \end{pmatrix}. \tag{4.35}$$

Therefore the gauge fields a_μ can be decomposed into "up/down" polarisations. With the ansatz $a_\mu = \varepsilon_\mu \varphi_{m,n}$ one easily obtains the V_{04} spectrum for the gauge fields

$$\begin{aligned} \omega_\uparrow^{2\pm} &= \frac{2m+1}{\theta_1} + \frac{2n+1}{\theta_2} \pm \frac{2}{\theta_1} + \mu^2\,, \qquad & a_{\mu\uparrow}^{\pm} &= \varphi_{m,n}\begin{pmatrix} \vec{\varepsilon}^{\,\pm} \\ 0 \end{pmatrix}, \\ \omega_\downarrow^{2\pm} &= \frac{2m+1}{\theta_1} + \frac{2n+1}{\theta_2} \pm \frac{2}{\theta_2} + \mu^2\,, \qquad & a_{\mu\downarrow}^{\pm} &= \varphi_{m,n}\begin{pmatrix} 0 \\ \vec{\varepsilon}^{\,\pm} \end{pmatrix}, \end{aligned} \tag{4.36}$$

where $\varphi_{m,n}$ is given in (4.33) and the polarisation vectors are $\vec{\varepsilon}^{\,\pm} = (i, \pm 1)$. Thus, depending on the polarisation the mode energies are shifted compared to the scalar field fluctuations (4.33). The eigenvalues of the V_{04} sector for the case that the scalar field background is trivial, which means $\mu = 0$ here, have been obtained in [26]. For a non-vanishing scalar field background, i.e. $\mu^2 > 0$, the mass of the V_{04} fluctuations increases with μ^2. On the other hand μ is the amplitude of the scalar field (4.27). This fits perfectly with the string interpretation of $\mathcal{X}_1$ as the separation of the $D0$- from the $D4$-brane. As the distance increases the energy for strings stretched between them also increases.

If we set $\mu = 0$ and use the identification (2.12) the low-lying mode energies of the V_{04} sector, i.e. $m, n = 0$ in (4.33) and (4.36), coincide with the energies obtained from quantising open strings stretched between the $D0$- and $D4$-brane [1]. As mentioned above we consider the regime $\theta_1 > \theta_2$ such that the tachyon sits in the "down" fluctuation:

$$\omega_\downarrow^{2-} = \frac{1}{\theta_1} - \frac{1}{\theta_2} + \mu^2 < 0 \qquad \text{if } \frac{\mu^2}{2} < \frac{\xi_1 - \xi_2}{\xi_1 \xi_2}\,. \tag{4.37}$$

For later convenience we have again introduced the quantities $\xi_\alpha := 2\theta_\alpha$. As we will see later on $\sqrt{\xi_1 - \xi_2}$ is the size of the instanton for non-self-dual $\theta_{\mu\nu}$. Thus if the $D0$-brane approaches the $D4$-brane below the "relative" instanton size $\frac{\xi_1 - \xi_2}{\xi_1 \xi_2}$ the system becomes unstable. Before commenting on the rest of the spectrum let us impose the gauge condition associated to the gauge-fixing action (4.24). In doing so one can see that the gauge field and scalar field zero-modes

$$\sim \varepsilon_\mu P\,, \qquad \sim \alpha_m P \;\in V_{00} \tag{4.38}$$

are unaffected by the gauge condition. The zero-modes in the gauge field fluctuations can be understood as the usual translational zero-modes associated to the position of the $D0$-brane or its projection in the $D4$-brane. But note that translations in noncommutative directions play a subtle role since they are equivalent to gauge transformations [3]. The zero-modes in the scalar fields are additional possible zero-mode moduli as were discussed in (4.22). Thus they refer to translations of the $D0$-brane in directions transversal to the $D4$-brane. They have no counterpart in the commutative U(2) instanton, where the operator $-\mathscr{D}^2$ is strictly positive (note that for self-dual $\theta_{\mu\nu}$ the background is self-dual). This is in one-to-one correspondence with the irreducibility of the field strength, i.e. that $F\chi = 0$ has no non-trivial solution [21]. Obviously the solution (4.12) does not satisfy this condition. This may also have some impact on index theorems on noncommutative space. In [27] for example it was assumed that the operator $\mathscr{D}^2$ has no zero-modes.

The gauge condition for the continuous V_{44} spectrum restricts the polarisations to be transversal to the momenta, i.e. $-\omega\varepsilon_0 + \varepsilon_\mu k_\mu = 0$. For the $V_{04,40}$ spectrum the gauge condition implies various relations between the expansion coefficients for these modes.

4.3 Self-dual point and T-duality

Let us have a closer look at the special point $\theta_1 = \theta_2 := \theta$ in the parameter space where the background gauge field (4.12) becomes self-dual and thus describes an instanton. The size of the instanton $\sqrt{\xi_1 - \xi_2}$ shrinks to zero in this limit. At the self-dual point of $\theta_{\mu\nu}$ the solution (4.12) thus describes a small instanton. Although the moduli space still has a small instanton singularity at this point, due to noncommutativity the solution (4.12) is regular for self-dual $\theta_{\mu\nu}$. The spectrum in the self-dual point can also be obtained continuously by the limit $\theta_{1,2} \to \theta$ from the spectra (4.33), (4.36). The states do not change except for those in the V_{44} sector, which become isotropic noncommutative plane waves. The mode energies of the $V_{04,40}$ sector are changed as follows:

$$\begin{aligned} \omega^2 &\to \frac{2}{\theta}\,(m+n+1) + \mu^2\,, \\ \omega_\uparrow^{\pm 2} &\to \frac{2}{\theta}(m+n+1\pm 1) + \mu^2\,, \\ \omega_\downarrow^{\pm 2} &\to \frac{2}{\theta}(m+n+1\pm 1) + \mu^2\,. \end{aligned} \tag{4.39}$$

The most important change in the spectrum is that now for arbitrary small separation $\sim \mu$ of the D-branes the spectrum is positive. The tachyon disappears. This was expected since the gauge field background is self-dual. This is consistent with the fact that the $D0$-$D4$ system is supersymmetric and BPS for (anti)self-dual B-field backgrounds [1].

The up/down gauge field modes are degenerate because of the enhanced symmetry. If we set $\mu = 0$, i.e. when the $D0$-brane sits on the top of the $D4$-brane, in addition to the translational zero-modes (4.38) two independent zero-modes in the $V_{04,40}$ sector appear. It is easy to see that they are not affected by the gauge condition. We do not have an explicit interpretation in terms of moduli for these additional zero-modes. We can only suppose that they are related to the "superconformal zero-modes". In [27] it was shown that for self-dual noncommutative instantons with self-dual background $\theta_{\mu\nu}$ these zero-modes are unaffected by the noncommutativity. The reason for this is that the ADHM constraints are the same as for the commutative case.

In the following we will assume that $\mu = 0$.

The fluxon and T-duality. In [3] the so called fluxon solution to equation (3.3) and its spectrum were considered. The corresponding D-brane picture is a tilted $D1$-brane piercing a $D3$-brane at an angle $\tan\psi = \frac{2\pi\alpha'}{\theta}$. The one-oscillator Fock space conventions are obtained by setting $\theta_1 = \theta$ and $\theta_2 = 0$. The single fluxon is given by ($D_z =: D,\ \bar{D}_{\bar{z}} =: \bar{D}$)

$$D = \frac{1}{\sqrt{\xi}} S c^\dagger S^\dagger \,, \qquad \bar{D} = -\frac{1}{\sqrt{\xi}} S c S^\dagger \,, \qquad \mathcal{A}_3 = 0 \,, \qquad \phi = -\frac{2i}{\xi} x^3 P \,, \tag{4.40}$$

where the shift operator S and the projector are now given by

$$S = \sum_{n=0}^{\infty} |n+1\rangle\langle n| \,, \qquad P = |0\rangle\langle 0| \,. \tag{4.41}$$

Introducing $\mathcal{A}_\mu = (\mathcal{A}_{i=1,2,3},\, \phi)$ as suggested in (3.3) one finds the associated field strength $\mathcal{F}_{\mu\nu}$ to be of the same form as the field strength (4.12) but with self-dual $\theta_{\mu\nu}$ and the projector as given in (4.41).

The fluctuation spectrum of the fluxon, analogously to the $D0$-$D4$ system, decomposes into different sectors V_{11}, V_{33}, $V_{13,31}$ characterising strings living on the $D1$- or the $D3$-brane and strings stretched between them. Analogously to (4.33) the scalar field eigenmodes X_m, except for fluctuations of the nontrivial Higgs field ϕ in (4.40), are eigenmodes of $-\mathscr{D}^2$ $(m, n \geq 0)$:

$$\begin{aligned}
&V_{11} : \omega^2 = k_3^2 \,, && \varphi_0 = e^{ik_3 x^3} P \\
&V_{13} : \omega^2 = \frac{2}{\theta}(m+n+1) \,, && \varphi_{mn} = e^{-x_3^2/\xi} H_m\left(\frac{x_3}{\sqrt{\theta}}\right) |0\rangle\langle n| S^\dagger \\
&V_{33} : \omega^2 = \vec{k}^2 \,, && \varphi_k = e^{ik_3 x^3} S e^{i\sqrt{\xi}(kc + \bar{k}c^\dagger)} S^\dagger \,.
\end{aligned} \tag{4.42}$$

Here H_m are the Hermite polynomials and we have omitted proper normalisation constants here (see [3] for details). The V_{11} fluctuation becomes a normalisable zero-mode (per unit length) for vanishing momentum k_3, describing translation moduli transverse to the $D1$-string [3].[8] They are analogous to zero-modes that we found in the V_{00} sector of the $D0$-$D4$ system (4.33). The $V_{13,31}$ spectrum is the same as the $V_{04,40}$ spectrum at the self-dual point, i.e. the first line in (4.39). As for the $D0$-$D4$ system there is a massless continuum in the $D1$-$D3$ system but the degeneracy is different.

Grouping also the gauge field and Higgs field fluctuations η as $a_\mu = (a_{i=1,2,3}, \eta)$ one obtains the fluctuations analogously to (4.36). As mentioned above, the background field strength $\mathcal{F}_{\mu\nu}$ is of the same form as for the instanton at the self-dual point and thus the fluctuation operator for the modes a_μ is of the form (4.34) with $\theta_1 = \theta_2 = \theta$. Hence the eigenmodes are decomposed according to the same polarisations as in (4.36). The mode energies are given by (4.39).

The equivalence of the spectra of the $D1$-$D3$ system and the $D0$-$D4$ system at the self-dual point could probably be understood from T-duality. By T-dualising the direction along the $D1$-brane the $D1$-brane becomes a $D0$-brane whereas the in this respect tilted

[8] Strictly speaking these zero-modes are not normalisable because of the x_3-integration. But having in mind that the $D1$-string will be T-dualised the motion along the $D1$-string is constrained to a point and so is its projection on the x_3 axis. Thus the plane wave in the x_3 direction might become a normalisable state as the one in the V_{44}-sector. See the following discussion.

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$D3$-brane turns into a $D4$-brane with and additional magnetic field on it. One thus ends up with a $D0$-$D4$ system with a rank four B-field. Hence, in the presence of a B-field T-duality transformations also act on the B-field background [1]. This way a noncommutative theory can be mapped to a commutative or partial commutative theory as we observe it here, where $\mathbb{R}^4_\theta \to \mathbb{R}^2_\theta \times \mathbb{R}$. At the classical level we observe that the number of zero-mode moduli is the same in both cases.[9] On the other hand the so called Nahm-duality identifies the moduli spaces of self-dual solutions on dual four-tori [28−30]. Different limits for the radii of the tori lead to different non-compact spaces as for example $\mathbb{R}^4$. This duality is also reflected in the ADHM construction of instantons (see below).

Here one finds similar identifications when T-dualising single directions and not only for the moduli space of self-dual configurations but also for the fluctuation spectrum. We will comment on this later when considering the ADHM construction, but we have to leave a more detailed investigation of this issue for future studies.

Supersymmetry of the spectrum of BPS backgrounds. We briefly discuss a property of the fluctuation spectrum in the presence of a self-dual background which will become important below. Introducing the generators of the quarternionic algebra, $\sigma^\mu = (\vec{\sigma}, -i\mathbb{1})$ and $\bar{\sigma}^\mu = (\vec{\sigma}, i\mathbb{1})$, one can define the self-dual and anti-self-dual projector

$$\sigma^{\mu\nu} := \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\,, \qquad \bar{\sigma}^{\mu\nu} := \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\,. \tag{4.43}$$

Writing the background covariant derivatives in spinor notation, $\not{D} := \sigma^\mu D_\mu$ and $\bar{\not{D}} := \bar{\sigma}^\mu D_\mu$, one has for a self-dual background the following two quadratic operators:

$$\not{D}\bar{\not{D}} = \mathscr{D}^2 + \sigma^{\mu\nu}[\mathcal{F}_{\mu\nu}, .\,]\,, \quad \bar{\not{D}}\not{D} = \mathscr{D}^2\,, \tag{4.44}$$

where we have used that $\bar{\sigma}^{\mu\nu}\mathcal{F}_{\mu\nu} = 0$ for a self-dual background. The factorisation property (4.44) implies that also on the noncommutative space the two operators in (4.44) are isospectral except for zero-modes. But as in the commutative case the spectral densities of the continuous spectra are in general different.[10]

Except for the fact that they are matrices in the two-dimensional spinor space the operators (4.44) are very similar to the fluctuation operators appearing in the quadratic action (4.28). This connection becomes explicit when writing the gauge field fluctuations in spinor space, i.e. $\not{a} = \sigma^\mu a_\mu$ and $\bar{\not{a}} = \sigma^\mu \bar{a}_\mu$. The lagrangian density for the gauge field fluctuations (4.28) then becomes:[11]

$$a_\mu(\delta^{\mu\nu}\mathscr{D}^2 + 2[\mathcal{F}^{\mu\nu}, .\,])a_\nu = \frac{1}{2}\,\mathrm{tr}\{\bar{\not{a}}\not{D}\bar{\not{D}}\not{a}\}\,, \tag{4.45}$$

where tr is the trace in spinor space and $a_\mu = \frac{1}{2}\,\mathrm{tr}\{\bar{\sigma}_\mu\not{a}\}$. Later we will make use of the isospectrality of the two operators (4.44) in computing the spectrum for the much simpler

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[9] There are five scalars X_m in both cases.

[10] In general we include the continuous spectrum in the term "spectrum". If necessary we explicitly distinguish between discrete and continuous spectrum.

[11] The analogous relation for the scalar fields is trivial since $\mathscr{D}^2$ in (4.44) is proportional to the unit in spinor space.

operator $\mathscr{D}^2$ without the additional spin-coupling. The states in question are then given by the non-zero eigenstates $\tilde{a}_\mu$ of $-\mathscr{D}^2$:

$$\not{a}_{(\omega)} = \frac{1}{\sqrt{\omega}} \not{D}\, \tilde{\not{a}}_{(\omega)} \tag{4.46}$$

This will considerably simplify the analysis when considering less trivial self-dual backgrounds for non-self-dual $\theta_{\mu\nu}$. Zero-modes of the first operator in (4.44) have to be treated separately.

4.4 Matrix models

Above we briefly mentioned the connection between the zero-modes (4.7) and matrix models. Since we now have discussed the fluctuation spectrum we can give a more detailed account on this relation. We restrict this discussion to self-dual $\theta_{\mu\nu}$ so that the background is supersymmetric and BPS. For convenience we adopt the ten dimensional language such that $A_M = (A_0, A_\mu, X_m)$. We are interested in the effective theory of the $D0$-$D0$-strings in a charge k background (4.12). The charge k background is characterised by a projection operator P_k which defines a subspace $V_k \otimes V_k^*$ (4.17). As we have learned from the spectrum (4.33) the $D0$-$D0$ fluctuations live in this subspace. According to (4.18) and (4.19) we decompose the ten-dimensional gauge field as follows:

$$A_M \to \mathcal{A}_M + a_M \qquad \text{with } a_M = \sum_{i,j=0}^{k-1} (a_M)^{ij} |i\rangle\langle j| \,. \tag{4.47}$$

The background $\mathcal{A}_M$ is nontrivial only for $M = \mu$ and given by (4.12). Analogously we expand the fermionic $D0$-$D0$ fluctuations according to:

$$\lambda = \sum_{i,j=0}^{k-1} \lambda^{ij} |i\rangle\langle j| \,, \tag{4.48}$$

so that they also solve the fermionic fluctuation equation with zero eigenvalues, $\Gamma^M D_M \lambda = 0$.

In the subspace $V_k \otimes V_k^*$ the connection D_μ acts trivially and the field strength $\mathcal{F}_{\mu\nu}$ is proportional to the unit. Therefore one obtains by inserting (4.47) in the action (4.1) the following action for the $D0$-$D0$ fluctuations:

$$S_{D0} = -\int dt \left[\frac{4\pi^2 k}{g_{YM}^2} + \frac{4\pi^2\theta^2}{2g_{YM}^2} \operatorname{Tr}_{\mathrm{U}(k)} \left\{ (\nabla_0\, a_I)^2 - \frac{1}{2}[a_I, a_J]^2 - 2(\lambda^T \nabla_0 \lambda + \lambda^T \Gamma^I [a_I, \lambda]) \right\} \right], \tag{4.49}$$

where $\nabla_0 = \partial_t + a_0$ and $I = 1, \ldots, 9$. Thus up to an irrelevant additive constant, one obtains the BFSS matrix model [22] where the coupling or tension is given by[12]

$$T_0 = \sqrt{2\pi}/g = \frac{4\pi^2\theta^2}{g_{YM}^2} \,. \tag{4.50}$$

[12] We follow the notation of [23].

It describes the dynamics of k $D0$-branes by a supersymmetric $\mathrm{U}(k)$ quantum mechanics obtained by the dimensional reduction of ten-dimensional commutative $\mathcal{N}=1$ $\mathrm{U}(k)$ super Yang-Mills theory to $0+1$ dimensions. In the $k=\infty$ limit the matrix model is conjectured to be equivalent to 11-dimensional M-theory [22].

A different relation between matrix models and noncommutative supersymmetric Yang-Mills theory was given in [31, 32]. The so-called $\mathrm{U}(N)$ IKKT matrix model [33] is the dimensional reduction of ten-dimensional $\mathcal{N}=1$ $\mathrm{U}(N)$ super Yang-Mills theory to zero dimensions. In the $N=\infty$ limit it describes noncommutative super Yang-Mills theory by excitations around nontrivial (D-brane) solutions of the IKKT matrix model. Thus in this case the mapping between matrix model and noncommutative super Yang-Mills theory is in the opposite way to the relation described above.

In this sense noncommutative super Yang-Mills theory plays a dual role. Its low energy dynamics in instanton backgrounds describing D-branes is a matrix model. At the same time it describes the excitations of D-brane solutions of this matrix model.

5. Instanton backgrounds with non-self-dual $\theta_{\mu\nu}$

We now consider self-dual solutions in a more general B-field background. In general these instantons will not be isometric to the vacuum as was the solution (4.13). In complex coordinates as given above (2.13) the self-duality equations (3.2) become

$$\mathcal{F}_{z_1\bar{z}_1}=\mathcal{F}_{z_2\bar{z}_2}\,,\qquad \mathcal{F}_{z_1\bar{z}_2}=0\,. \tag{5.1}$$

The holomorphic structure of the self-duality equations becomes manifest if one relabels the second coordinate $z_2\leftrightarrow\bar{z}_2$. But we prefer to keep the correspondence $D_{z_\alpha}\sim c_\alpha^\dagger$ for both coordinates. See also footnote 2 on page 6. Expressed through the operators (2.17) the equations (5.1) read

$$[D_{z_1},\bar{D}_{\bar{z}_1}]-[D_{z_2},\bar{D}_{\bar{z}_2}]=\frac{1}{\xi_1}-\frac{1}{\xi_2}\,,\qquad [D_{z_1},\bar{D}_{\bar{z}_2}]=0\,. \tag{5.2}$$

On the commutative $\mathbb{R}^4$ there exists a systematic way to construct all possible solutions to the self-duality equations (3.2). This is the the so-called ADHM-construction [34], which can be understood by Nahm-duality. In [14] it was shown that this method can be extended to noncommutative $\mathbb{R}^4_\theta$ by adding a F(ayet)I(liopoulos)-term to the ADHM constraints. In [35] the Nahm construction of monopoles was also extended to $\mathbb{R}^2_\theta\times\mathbb{R}$. In section 4.3 we observed a duality in the spectrum of the fluxon, which is a certain limit of the monopole, and the self-dual solution (4.12), i.e. for self-dual $\theta_{\mu\nu}$. To make the above statements more explicit and to be able to identify the tachyon mode with certain terms in the instanton solution for non-self-dual $\theta_{\mu\nu}$ we briefly go through the ADHM construction of noncommutative instantons. We follow here the representation of [28]. As mentioned above, we consider the (self-dual) regime $\theta_1\geq\theta_2>0$.

ADHM construction. The main input for the ADHM construction are the ADHM data. They collect the moduli of the general solution. For a $\mathrm{U}(N)$ k-instanton they are given by matrices $B_1, B_2 : V \mapsto V$ and $I : W \mapsto V,\ J : V \mapsto W$ with the vector spaces $V \cong \mathbb{C}^k, W \cong \mathbb{C}^N$. These matrices enter the ADHM constraints in the following way: Looking for a self-dual solution one introduces the zero-dimensional "Dirac operator"

$$\mathcal{D} = a + b(\sigma^\mu x_\mu \otimes \mathbb{1}_k)\,, \tag{5.3}$$

where the x_μ are now noncommutative and the matrices a, b are given by

$$a = \begin{pmatrix} -B_2^\dagger & B_1 \\ B_1^\dagger & B_2 \\ I^\dagger & J \end{pmatrix}, \qquad b = \begin{pmatrix} \mathbb{1}_{2k} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.4}$$

The matrix σ^μ is given above (4.43). If one considers anti-self-dual solutions one has to take $\bar{\sigma}^\mu$ instead of σ^μ. Self-duality requires now that $\mathcal{D}^\dagger\mathcal{D}$ commutes with σ^μ. This gives the ADHM constraints

$$\mu_r := [B_1, B_1^\dagger] + II^\dagger - J^\dagger J = \xi_2 - \xi_1\,, \qquad \mu_c := [B_1, B_2] + IJ = 0\,. \tag{5.5}$$

Here μ_r, μ_c are the so called moment maps and are related to the hyper-Kähler construction of manifolds. The right hand side of the μ_r equation is due to the noncommutativity of the coordinates x^μ. This is the FI-term as it was added in [14]. For $\theta_{\mu\nu}$ being in the self-dual regime, i.e. $\xi_{1,2} > 0$, the FI-term vanishes at the self-dual point. This is different from the case considered in [14, 15], where the duality of the B-field was opposite to the duality of the gauge field, so that the FI-term is always nonzero. The moduli $B_{1,2}$ correspond to the collective coordinates and the moduli I, J to the size of the (multi) instantons. If the group is U(1) the self-dual point $\xi_1 = \xi_2$ corresponds to the small instanton singularity of the moduli space. For generic B-field, i.e. for $\xi_1 \neq \xi_2$, this singularity is resolved.

Next one has to solve the zero-dimensional "Dirac equation" and conditions

$$\mathcal{D}^\dagger\Psi = 0\,, \qquad \Psi^\dagger\Psi = 1\,, \qquad \Psi\Psi^\dagger + \mathcal{D}\frac{1}{\mathcal{D}^\dagger\mathcal{D}}\mathcal{D}^\dagger = \mathbb{1}\,. \tag{5.6}$$

The self-dual connection is then given by

$$D_\mu = i\Psi^\dagger\theta^{-1}_{\mu\nu}x^\nu\Psi\,. \tag{5.7}$$

The second relation in (5.6) is just a normalisation condition, where the last equation is a completeness relation, which is trivially fulfilled in commutative space but not in noncommutative space [13, 36−39].[13] This condition will be crucial for the interpretation of our solution.

Nahm- and T-duality. The appearance of the zero-dimensional "Dirac equation" in the above construction has its origin in Nahm-duality. The Nahm transformation maps the moduli space for U(N) k-instantons on the four-torus T^4 to the moduli space of U(k) N-instantons on the dual torus $\hat{T}^4$:

$$\mathcal{M}^{k,N}_{T^4} \simeq \mathcal{M}^{N,k}_{\hat{T}^4}\,. \tag{5.8}$$

[13] We are grateful to Kirsten Vogeler for pointing this out to us.

By sending the dual radii to zero the original four torus approaches $\mathbb{R}^4$ or $\mathbb{R}^4_\theta$. Thus the dual equations are zero-dimensional and therefore can be solved for particular cases. The original connection is then given by the Nahm transformation (5.7). On the string theory side the corresponding duality is T-duality. It maps the moduli space of k $D0$-branes sitting on a stack of N $D4$-branes wrapped around a four-torus T^4 to the moduli space of N $D0$-branes on k $D4$-branes wrapped around the dual torus $\hat{T}^4$.

A similar construction, the so called Nahm construction, exists also for monopoles on $\mathbb{R}^3$ or $\mathbb{R}^2_\theta \times \mathbb{R}$ [40, 41, 35]. For this one shrinks one radius of T^4 to zero and sends the other ones to infinity. The dual equations are thus one-dimensional (ordinary) differential equations. The Nahm transformation then gives the monopole gauge field and the nontrivial Higgs field. The corresponding brane picture for a U(2) monopole is a $D1$-brane/string stretched between two separated $D3$-branes [42]. The fluxon is obtained by sending one of the $D3$-branes to infinity. Thus the distance between them and therefore the mass of the massive fields become infinite. The massive fields decouple and only the massless U(1) fields remain [3].

Now T-duality is a more general concept than Nahm transformations tailored for self-dual connections. As discussed before, by T-dualising along the D-string which pierces the remaining $D3$-brane with a rank two B-field on it one obtains a $D0$-$D4$ system with a rank four B-field. T-duality also acts on the string fluctuations on the branes. So in principle one should find this duality also for the quantum fluctuations of the low-energy effective descriptions of these brane systems, at least for that part of the spectrum that survives the zero-slope limit. Therefore we think that the equivalence of the fluctuation spectra of the fluxon and the instanton for self-dual $\theta_{\mu\nu}$ that we found before is due to the T-duality in the D-brane description. The classical action/energy of the two objects is different but this is not surprising, since T-duality changes the charges of D-branes. As mentioned above we have to leave a more detailed description for future studies.

The single U(1) instanton. In the following we concentrate on the case[14] $k = N = 1$. For simplicity we also set the translational moduli $B_{1,2}$ in (5.5) to zero. So for $\xi_1 > \xi_2$

$$|J|^2 = \xi_1 - \xi_2 \,, \qquad I = 0 \tag{5.9}$$

solves (5.5). Now we can see that $|J| = \sqrt{\xi_1 - \xi_2}$ is the size of the instanton and therefore the instanton shrinks to zero at the self-dual point.[15] Following the steps described in (5.6), (5.7) one obtains the following solution (from now on we use complex coordinates if not stated differently and we abbreviate D_{z_α} with D_α):

$$\begin{aligned} D_1 &= \frac{1}{\sqrt{\xi_1}} S\Lambda^{\frac{1}{2}}\, c_1^\dagger\, \Lambda^{-\frac{1}{2}} S^\dagger \,, \\ D_2 &= \frac{1}{\sqrt{\xi_2}} S\Lambda^{-\frac{1}{2}}\, c_2^\dagger\, \Lambda^{\frac{1}{2}} S^\dagger + \frac{J}{\sqrt{\xi_1 \xi_2}} SP \,, \end{aligned} \tag{5.10}$$

[14] For noncommutative U(2) multi-instanton configurations see e.g. [13, 43, 36, 44, 37].

[15] The notion of "size" has to be taken with care in noncommutative space.

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where the shift operator $S = S^{\mathrm{II}}$ and the projection operator P are defined in (4.9) and (4.26). The operator Λ is given by

$$\Lambda = \frac{M+\xi_1}{M+\xi_2}\,, \qquad M = \xi_1 n_1 + \xi_2 n_2\,, \tag{5.11}$$

where n_α are the number operators of the two Hilbert spaces (2.15). The adjoint connections are given by $\bar{D}_\alpha = -D_\alpha^\dagger$. The solution (5.10) is by construction a self-dual charge one field configuration. Thus its five brane mass according to (4.14) is given by

$$M_5^{BPS} = \frac{4\pi^2}{g_{YM}^2}\,. \tag{5.12}$$

The instanton (5.10) differs in two respects from the instanton at the self-dual point $\xi_1 = \xi_2$ (4.13). First there is an additional "axial" term proportional to the size J of the instanton. This term is crucial for the completeness relation in (5.6). As we will see below it has its origin in the tachyonic mode that we found for the unstable solitonic state (4.37). Second the creation/annihilation operators are dressed by the operators Λ, which "entangle" the two Hilbert spaces. In the self-dual limit $\xi_2 \to \xi_1$ the solution (5.10) smoothly approaches the solution (4.13) at the self-dual point.

5.1 Fluctuation spectrum

As we show in the next section the instanton (5.10) will be the final state of the tachyon condensation process of the unstable configuration (4.12). To characterise this final state in more detail we investigate the fluctuation spectrum, i.e. the asymptotic quantum states in the presence of the background (5.10). The background is now much more complicated than before and it does not seem possible to solve the fluctuation equation for the gauge field modes. But as we have shown above, also in the noncommutative case the gauge-field fluctuation operator is, except for zero-modes, isospectral to the much simpler operator $\mathscr{D}^2$ (4.44). Possible zero-modes will be treated separately. Thus we are looking for solutions of

$$-\Delta\varphi_\omega = \omega^2\varphi_\omega \qquad \text{with} \;\; \Delta = 2\sum_{\alpha=1,2}\big([D_\alpha,[\bar{D}_\alpha,.\,]] + [\bar{D}_\alpha,[D_\alpha,.\,]]\big)\,, \tag{5.13}$$

where D_α is given by (5.10). Because of the axial term in (5.10) there are no longer any finite-dimensional invariant subspaces. Nevertheless it will prove to be useful to decompose the operator φ_ω as before according to the sectors $V_{00}, V_{04,40}$ and V_{44} (4.30). Let us now specify the different terms in (4.30) as follows:

$$\begin{aligned} a &= \sum_{m,n} a_{mn}|0,0\rangle\langle m,n|\,, \qquad b = \sum_{m,n} b_{mn}|m,n\rangle\langle 0,0|\,, \\ d &= \sum_{m_\alpha,n_\alpha} d^{\;m_1,m_2}_{\;n_1,n_2}\;|m_1,m_2\rangle\langle n_1,n_2|\,, \quad \alpha\in\mathbb{C}\,. \end{aligned} \tag{5.14}$$

The index α labels the two noncommutative planes described by the two parameters $\xi_\alpha = 2\theta_\alpha$ (not to be confused with the coefficient $\alpha \in \mathbb{C}$ in (4.30)). If not stated differently summations always run from zero to infinity. Introducing the abbreviations

$$\begin{aligned} Z_1^\dagger &:= \Lambda^{\frac{1}{2}}\ c_1^\dagger\ \Lambda^{-\frac{1}{2}} = \sum_{p,q} A_{p,q} |p+1,q\rangle\langle p,q|\,, \\ Z_2^\dagger &:= \Lambda^{-\frac{1}{2}}\ c_2^\dagger\ \Lambda^{\frac{1}{2}} = \sum_{p,q} B_{p,q} |p,q+1\rangle\langle p,q| \end{aligned} \tag{5.15}$$

for the dressed creation operators and $\rho := \frac{J}{\sqrt{\xi_1\xi_2}}$ for the "strength" of the axial part in (5.10) the fluctuation equation (5.13) in the different sectors are of the form:

$$\begin{aligned} V_{00} &: \alpha(4|\rho|^2 - \omega^2) - 4|\rho|^2 d^{0,0}_{0,0} = 0\,, \\ V_{04} &: a_{mn}(2y|\rho|^2 - \omega^2 + 2N_{mn}) - \frac{4\bar{\rho}}{\sqrt{\xi_2}} B_{mn} d^{0,0}_{m,n+1} = 0\,, \\ V_{40} &: b_{mn}(2y|\rho|^2 - \omega^2 + 2N_{mn}) - \frac{4\rho}{\sqrt{\xi_2}} B_{mn} d^{m,n+1}_{0,0} = 0\,, \end{aligned} \tag{5.16}$$

where $y = 2$ for $\{m,n\} = \{0,0\}$ and equal to one otherwise. We also introduced the abbreviation

$$N_{mn} := \frac{1}{\xi_1}(A^2_{m-1,n} + A^2_{mn}) + \frac{1}{\xi_2}(B^2_{m,n-1} + B^2_{mn})\,. \tag{5.17}$$

The V_{44}-sector equation is more involved:[16]

$$\begin{aligned} &2\sum_{\alpha=1,2} \frac{1}{\xi_\alpha}\Big([Z_\alpha,[Z_\alpha^\dagger,d]] + [Z_\alpha^\dagger,[Z_\alpha,d]]\Big) - \omega^2 d - \\ &-\frac{4\rho}{\sqrt{\xi_2}}\sum_{m,n} a_{m,n-1}B_{m,n-1}|00\rangle\langle mn| - \frac{4\bar{\rho}}{\sqrt{\xi_2}}\sum_{m,n} b_{m,n-1}B_{m,n-1}|mn\rangle\langle 00| - \\ &-4|\rho|^2\alpha|00\rangle\langle 00| + 2|\rho|^2\sum_{m,n}\Big(d^{0,0}_{mn}|00\rangle\langle mn| + d^{mn}_{0,0}|mn\rangle\langle 00|\Big) = 0\,. \end{aligned} \tag{5.18}$$

By inspection of the equations (5.16), (5.18) one can see that the axial term in the connection (5.10) introduces a mixing of the different sectors in a very special way. The equations (5.16) fix the V_{44} operator d at the "boundaries" of the two noncommutative planes, whose Hilbert space descriptions are parametrised by two lattices $\{m_\alpha, n_\alpha\}$ with $m_\alpha, n_\alpha \geq 0$. Thus the solution for the eigenvalue equation is obtained by implementing these boundary conditions on the general solution of the *d-equation*

$$H\ d = \epsilon^2\ d\,, \tag{5.19}$$

where this is essentially the first line in (5.18) which also defines the operator H. The boundary conditions then fix the mode energies ω^2. Let us assume for the moment that

[16] Here and in the following, terms with negative indices are not present. Equivalently they can be put to zero whenever such factors formally appear in an equation.

the coefficients of α, a_{mn} and b_{mn} in (5.16) do not vanish. We will comment on the opposite case below. Then using (5.19) equation (5.18) can be written as

$$(\epsilon^2-\omega^2)d = \frac{16|\rho|^2}{\xi_2}\sum_{m,n}\frac{B^2_{m,n-1}}{2y|\rho|^2-\omega^2+2N_{m,n-1}}\left(d^{00}_{m,n}|00\rangle\langle mn| + d^{m,n}_{00}|mn\rangle\langle 00|\right) + \\ +\frac{16|\rho|^4}{4|\rho|^2-\omega^2}\, d^{0,0}_{0,0}\, P - 2|\rho|^2\sum_{m,n}\left(d^{0,0}_{mn}|00\rangle\langle mn| + d^{mn}_{00}|mn\rangle\langle 0,0|\right). \tag{5.20}$$

Thus solving the problem (5.19) is crucial to get the fluctuation spectrum.

5.2 The disentangled system

The operators (5.15) have the same mapping property as the undressed creation operators in the sense that they leave the following subspaces of $\mathcal{H}\otimes\mathcal{H}^*$ invariant:

$$\Psi_{a,b} = \sum_{m,n} O_{mn}|m+a,n+b\rangle\langle mn|\,, \tag{5.21}$$

and analogously for other combinations of shifts in the occupation number. To understand the general structure of the (quantum) states in the presence of the instanton background we discuss a simplified version of (5.19). Let us "switch" off the entanglement by setting the operators Λ to the unit in (5.10). Thus equation (5.19) becomes the free equation (4.31). But the noncommutative plane waves (4.32) are an improper basis of solutions in this case. The boundary conditions imposed by (5.16) or (5.20), respectively, overconstrain the system since the noncommutative plane wave does not vanish on the whole boundary, except for zero momentum. The appropriate basis to fulfil these boundary conditions is of the form (5.21). Operators of the form (5.21) are non-vanishing only for one point at the boundary of the lattice $m_\alpha, n_\alpha \geq 0$, thus giving one condition to determine the single free parameter ω^2. For the simplified case $\Lambda = \mathbb{1}$ the equation (5.19) can easily be Weyl-transformed to the star-product representation by the inverse version of (2.9). Because of the vanishing gauge field for $\Lambda = \mathbb{1}$, H simply becomes the four-dimensional laplacian in the space of ordinary functions.

Discrete spectrum. The ansatz (5.21) is realized by polar coordinates r_α, φ_α for the two noncommutative planes. Therefore the general solution of (5.19) according to the ansatz (5.21) in the inverse Weyl-transformed version is given by

$$\frac{1}{2}\, e^{ia\varphi_1} J_a(k_1 r_1)\, e^{ib\varphi_2} J_b(k_2 r_2)\,, \qquad \epsilon^2 = k_1^2 + k_2^2\,, \tag{5.22}$$

where J_p is the order p Bessel function of the first kind [45], which are regular at the origin, and $k_1^2 = k_{x_1}^2 + k_{x_2}^2$, $k_2^2 = k_{x_3}^2 + k_{x_4}^2$ are the total momenta in the two planes. Normalisability of the states implies $k_{1,2} > 0$ and thus the spectrum is positive. The eigenstates according to the ansatz (5.21) are angular momentum eigenstates, with angular momentum quantum numbers a, b in the first/second noncommutative plane. Weyl-transforming (5.22) back to the operator representation one obtains the coefficients in (5.21) as follows:

$$O_{mn}(a,b,k) = \sqrt{\frac{m!n!}{(m+a)!(n+b)!}}\left(\frac{k_1\theta_1}{2}\right)^{a/2}\left(\frac{k_2\theta_2}{2}\right)^{b/2}\times$$

$$\times e^{-\frac{k_1^2\theta_1+k_2^2\theta_2}{4}} L_m^a\left(\frac{k_1^2\theta_1}{2}\right) L_n^b\left(\frac{k_2^2\theta_2}{2}\right), \tag{5.23}$$

where the L_m^a are the Laguerre polynomials [45]. As one can see, the boundary matrix element $O_{0,0}$ vanishes only for infinite momentum in one of the two planes, so that for asymptotic high momentum the states would no longer feel the instanton background. But for generic momentum one obtains from (5.20) the following conditions: The "bulk" matrix elements imply $\epsilon^2 = k^2 = \omega^2$ whereas from the boundary matrix element one obtains

$$\omega_{ab}^2 = 2\left(y|\rho|^2 + N_{a,b-1} - \frac{4}{\xi_2}\, B_{a,b-1}^2\right) \qquad \text{for } a, b \neq 0 \tag{5.24}$$

where $y = 2$ for $a = 0, b = 1$ and equal to one otherwise. The diagonal operator with $a, b = 0$ would be a zero-mode, i.e. $\omega_{0,0}^2 = 0$, but this mode is not normalisable and therefore has to be excluded. By hermiticity one obtains the same frequencies for the hermitian conjugate of the operator (5.21).

For the disentangled case, where $\Lambda = \mathbb{1}$, the elements N_{mn}, B_{mn} simplify to

$$N_{mn} = \frac{2m+1}{\xi_1} + \frac{2n+1}{\xi_2}, \qquad B_{mn} = \sqrt{n+1}, \tag{5.25}$$

so that with $|\rho|^2 = (1/\xi_2 - 1/\xi_1)$ we obtain for the mode energies

$$\omega_{ab}^2 = 4\left(\frac{a}{\xi_1} - \frac{b}{\xi_2}\right) \quad \text{with } b > 0\,. \tag{5.26}$$

The mode $\Psi_{0,1}$ is negative, i.e $\omega_{0,1}^2 = -2(1/\xi_1 + 1/\xi_2) < 0$, and therefore is also non-normalisable. But also for other values of a, b normalisability excludes some of the modes with energy (5.26). Positivity of the mode energies (5.26) requires

$$a > \frac{\xi_1}{\xi_2}\, b\,, \tag{5.27}$$

which, depending on the relative size of the noncommutative parameters, excludes a number of additional modes from the discrete spectrum. Note that we always have $\xi_1 > \xi_2$.

Continuous spectrum. In addition to the ansatz (5.21) one can find compatibility with the boundary terms in (5.18) for operators of the following form:

$$\Psi_{ab} = \sum_m O_{mn}|m+a, n\rangle\langle m, n+b| \quad \text{with } a, b \neq 0\,. \tag{5.28}$$

For this operator all boundary matrix elements that occur in the equations (5.16), (5.18) are zero. Thus a solution of (5.19) is automatically a solution of the whole system with $\alpha = a_{mn} = b_{mn} = 0$ if $\epsilon^2 = \omega^2$. It turns out that the coefficients O_{mn} in (5.28) are also given by (5.23). The eigenvalues are again given by

$$\epsilon^2 = \omega^2 = k_1^2 + k_2^2\,, \tag{5.29}$$

where we again use the notation as introduced in (5.22), but the momenta are now unconstrained except for the positivity condition $k_\alpha > 0$. Thus one also finds a continuous spectrum.

It can be easily seen, that if the coefficients of α, a_{mn}, b_{mn} in (5.16) vanish one does not obtain new solutions.

5.3 Entangled system

Let us now discuss how the above results will be modified if one takes the nontrivial Λ (5.11) into account. The equations (5.16), (5.18) as well as the invariant subspaces (5.21), (5.28) are unchanged, but the matrix elements B_{mn} and N_{mn} are less trivial now. Therefore also equation (5.19) becomes more complicated and the mode-functions will not be of the simple form (5.22) but will include nontrivial form factors, such that the solutions no longer factorise according to the two noncommutative planes. In operator language this means that the matrix elements O_{mn} will include these form factors and will also no longer factorise. So Ψ_{ab} can no longer be written as a sum of products of two operators. Therefore we use the notion (dis)entangled. But since the general structure is unchanged the spectrum again separates into a discrete part and a gap-less continuum as for the disentangled case. The relation for the discrete spectrum (5.24) also remains valid, but now one has to use in (5.17), (5.24) the nontrivial matrix elements

$$A^2_{mn} = (m+1)\frac{\Lambda_{m+1,n}}{\Lambda_{mn}}\,, \qquad B^2_{mn} = (n+1)\frac{\Lambda_{mn}}{\Lambda_{m,n+1}}\,, \tag{5.30}$$

where the matrix elements Λ_{mn} can be read off (5.11). The mode energies for the states Ψ_{ab} are now given by ($b > 0$):

$$\omega^2_{ab} = 4\left(\frac{a}{\xi_1} - \frac{b}{\xi_2} + \frac{a+b-1}{a\xi_1 + (b-1)\xi_2} - \frac{2(a+b)}{a\xi_1 + \xi_2} + \frac{a+b+1}{(a+1)\xi_1 + b\xi_2}\right). \tag{5.31}$$

As one can see the, mode energies (5.24) are now rather complicated functions of a, b, but they describe the exact spectrum in the background (5.10). Again one has to check if there are certain values a, b which have to be excluded from the spectrum. This time we do not have explicit solutions for the operators Ψ_{ab} so that we cannot explicitly exclude negative modes by normalisation conditions. But we expect that this property will not be changed by nontrivial form factors. Anyway, from the supersymmetry or factorisation property (4.44) of the fluctuation operators with self-dual classical background fields one knows that the spectrum has to be positive. Thus as before we exclude modes with $\omega^2_{ab} < 0$ from the spectrum. It is easy to see that $\omega^2_{ab} > 0$ if

$$a > \frac{\xi_1}{\xi_2}\, b + \epsilon\,, \tag{5.32}$$

where $\epsilon < 1$. Thus one obtains roughly the same condition as for the disentangled system. The diagonal operator Ψ_{00} would be a zero-mode, but as before we expect that this mode as the zero momentum threshold mode of the continuum is not normalisable, or equivalently, the field strength $\mathcal{F}_{\mu\nu}$ is no longer reducible. We will comment on this in the discussion below.

Zero-modes. So far we have investigated the operator $-\mathscr{D}^2$ which is except for zero-modes isospectral to the operator in question (4.44). Since this time $-\mathscr{D}^2$ has no zero-modes the kernel of the nontrivial operator in (4.44) and the kernel of $\bar{\not{D}}$ coincide:

$$\mathrm{kern}\{\bar{\not{D}}\} = \mathrm{kern}\{\not{D}\bar{\not{D}}\}\,. \tag{5.33}$$

But the number of bosonic zero-modes is twice the number of single solutions to $\bar{\not{D}}\chi = 0$ [21]. Since now $-\mathscr{D}^2$ is strictly positive the considerations of [27] are valid without any restrictions. Thus for a U(N) k-instanton one obtains $4Nk$ bosonic zero-modes. In our case these are four zero-modes corresponding to translations of the $D0$-brane inside the $D4$-brane, or more exactly, translations of the $D0$-$D4$-bound state object. Since for vanishing scalar fields $\bar{\not{D}}\chi = 0$ is a fermionic equation of motion in the euclidean four-dimensional theory a rather simple method to obtain solutions to it is via the broken supersymmetries in the instanton background. Thus one obtains the fermionic zero-modes as

$$\chi_\pm \sim \sigma^{\mu\nu}\mathcal{F}_{\mu\nu}\epsilon_\pm\,, \tag{5.34}$$

where $\epsilon_\pm$ are two linearly independent two component spinors. Hence the number of bosonic zeromodes is indeed four, i.e. $4Nk$ with $N = k = 1$.

5.4 Discussion of the spectrum

As a first difference to the spectrum for the small instanton, i.e. for self-dual $\theta_{\mu\nu}$, or the unstable state with non-self-dual $\theta_{\mu\nu}$, respectively we note that the operator $-\mathscr{D}^2$ has now no zero-modes. As we discussed above (4.38) these zero-modes correspond to translational moduli of the $D0$-brane(s) transverse to the $D4$-brane. Thus for the instanton solution away from the self-dual point (5.10) the $D0$-branes are confined in the $D4$-brane. This confirms the description as given in [43] for example. For a self-dual B-field the lower dimensional brane of a $Dp - D(p+4)$ system can freely move transverse to the higher dimensional brane and thus separate from the $D(p+4)$-brane. The effective theory on the lower dimensional brane is in the Coulomb phase. On the contrary for non-self-dual $\theta_{\mu\nu}$ there is no small instanton singularity and the lower dimensional brane is confined in the $D(p+4)$-brane. In this case the effective theory on the lower dimensional brane gains a FI-term and is thus in the Higgs phase. This is reflected by the axial term in the solution (5.10) living on the higher dimensional brane. This axial- or FI-term is also related to the appearance of the tachyonic mode (4.37) for the unstable configuration as we will see. Because of the mixing property of the axial term in the Hilbert space the identification of the $D0$-brane according to $(0,0)$, $(0,4)$ and $(4,4)$ fluctuations is no longer possible. In forming a $D0$-$D4$ bound state the $D0$-brane looses its strict individual character.

Second, half of the continuum states are tied up with the states of the discrete spectrum, such that the number or density of continuous states is reduced. Because of the condition (5.32) the number of discrete states is also reduced compared to the self-dual point. The eigenvalues for the discrete states (5.31) are exact.

The continuous and discrete states are angular momentum eigenstates w.r.t. the angular momenta in the two noncommutative planes. States with the same orientation of the angular momenta in the two planes (5.21) are discrete states, whereas states with opposite angular momentum orientations in the two planes (5.28) remain continuum states.

As expected the number of zero-modes in the gauge field fluctuations is unchanged. They describe the translational moduli of the $D0$-$D4$-bound state object.

One set of possible additional discrete states could not be investigated here. For entangled system, i.e. $\Lambda \neq \mathbb{1}$, the equation (5.19) could have additional bound states. To find them one would have to solve (5.19) explicitly, which is a considerably hard problem.

6. Tachyon condensation

In the last section we have constructed a (self-dual) BPS state (5.10) for non-self-dual $\theta_{\mu\nu}$ which takes values in the self-dual regime $\theta_1 > \theta_2 > 0$. We characterised this state also by its excitation spectrum. In this section we will argue that this state is the final state of the tachyon condensation process of the unstable $D0$-$D4$ system (4.13) which was described above. A first hint to the fact that the unstable state decays into the BPS state (5.10) and not to the vacuum, like unstable solitonic states in 2+1 dimensions [19], is given by energetic considerations. The BPS state (5.10) is a stable state and its energy (mass) (5.12) is below the energy of the unstable state (4.14). But we will give more convincing arguments below.

6.1 Relevant degrees of freedom

Let us first write the tachyonic mode in (4.36) in complex coordinates. The total tachyonic excitation of the gauge field is an anti-hermitian linear combination of the V_{04} and V_{40} sector tachyonic fluctuations. This way one obtains

$$a^{th}_{z_2} = t\; SP\,, \qquad \bar{a}^{th}_{\bar{z}_2} = -\bar{t}\; PS^\dagger\,, \qquad a^{th}_{z_1} = \bar{a}^{th}_{\bar{z}_1} = 0\,, \tag{6.1}$$

where t is the (complex) tachyon amplitude. Thus exciting the tachyonic mode on the unstable background (4.13) adds an operator of the form $|0,1\rangle\langle 0,0|$ to D_2. The idea for calculating the tachyon potential is to neglect the exact time evolution of the condensation process and to consider only the potential for static fields. The kinetic term remains unknown. By "integrating out" all degrees of freedom except the tachyon mode one obtains the potential as a function of the tachyon amplitude only, which is then the tachyon potential. Here "integrating out" means minimising the potential and inserting the solutions for the relevant degrees of freedom. Since we consider now only static configurations it is convenient to work in the temporal gauge $A_0 = 0$. For static configurations the associated Gauß law constraint is automatically fulfilled.

The initial state, i.e. the unstable solution (4.13) including the tachyonic excitation (6.1) as well as the final state (5.10) are operators of the form $\sum_{p,q} L_{pq}|p+1,q\rangle\langle p,q|$ or with shifted arguments in the second Hilbert space factor and of course the hermitian conjugates thereof. Thus following the symmetry arguments of [19] we make the following ansatz for the relevant degrees of freedom:[17]

$$\nabla_1 = \frac{1}{\xi_1} S Z_1^\dagger(t) S^\dagger\,, \qquad \nabla_2 = \frac{1}{\xi_2} S Z_2^\dagger(t) S^\dagger + tSP\,, \tag{6.2}$$

and $\bar{\nabla}_\alpha = -(\nabla_\alpha)^\dagger$. The operators $Z_\alpha^\dagger(t)$ are of the form

$$Z_1^\dagger(t) = \sum_{p,q} A_{p,q}(t)|p+1,q\rangle\langle p,q|\,, \qquad Z_2^\dagger(t) = \sum_{p,q} B_{p,q}(t)|p,q+1\rangle\langle p,q|\,. \tag{6.3}$$

[17] In the following we suppress the dependence on the complex conjugate amplitude $\bar{t}$ in our notation.

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The operators (6.3) should not be confused with the operators (5.15) but we will see that after the condensation process they will coincide. The ansatz (6.2), (6.3) implies that not only the initial and final state are of this form, but also during the condensation process only fields of this form will contribute. On this reasoning also the first positive excitation in (4.36) should have been included. It would add a term to the operator ∇_1 which is analogous to the tachyon mode. But as it turns out by integrating out the non-tachyonic degrees of freedom the amplitude of this mode vanishes. Putting the scalar fields and fermions to zero also satisfies the minimum condition for the potential.

Thus, inserting the ansatz (6.2) into the action (4.1), one obtains the following potential[18]

$$V = \frac{2}{g_{YM}^2} \int \mathrm{Tr} \left[\sum_\alpha \left([\nabla_\alpha, \bar{\nabla}_\alpha] - \frac{1}{\xi_\alpha} \right)^2 - \{[\nabla_1, \nabla_2], [\bar{\nabla}_1, \bar{\nabla}_2]\} - \{[\nabla_1, \bar{\nabla}_2], [\bar{\nabla}_1, \nabla_2]\} \right] . \tag{6.4}$$

Minimising the potential (6.4) w.r.t. $Z_\alpha(t)$ one obtains (suppressing the t-dependence in our notation) the following equations:

$$\begin{aligned} &[Z_1^\dagger, \mathcal{K}] - |t|^2 Z_1^\dagger P + \frac{2}{\xi_2}[Z_2^\dagger, [Z_1^\dagger, Z_2]] = 0 \,, \\ &[Z_2^\dagger, \mathcal{K}] - |t|^2 Z_2^\dagger P + \frac{2}{\xi_1}[Z_1^\dagger, [Z_2^\dagger, Z_1]] = 0 \,, \end{aligned} \tag{6.5}$$

where we have introduced the operator

$$\mathcal{K} = \frac{1}{\xi_1}[Z_1^\dagger, Z_1] - \frac{1}{\xi_2}[Z_2^\dagger, Z_2] \,, \tag{6.6}$$

and we have used that $Z_\alpha P = 0$ due to the ansatz (6.3). To find an exact solution to the nonlinear equations (6.5) is rather difficult. Note that in the 2+1 dimensional case, as it was considered in [19], the analogous equations become linear. The initial and final state fall into a certain class of solutions to (6.5) which are of the form

$$\mathcal{K}(t) = \frac{|J|^2}{\xi_1 \xi_2} + |t|^2 P \,, \qquad [Z_1^\dagger, Z_2] = 0 \,. \tag{6.7}$$

We can an only give an approximate solution to (6.5):

$$\begin{aligned} \langle pq|Z_1^\dagger|pq\rangle = A_{pq} &= \sqrt{p+1}\sqrt{1 - |\tau|^2 \left(1 - \frac{\Lambda_{p+1,q}}{\Lambda_{pq}}\right)} \\ \langle pq|Z_2^\dagger|pq\rangle = B_{pq} &= \sqrt{q+1}\sqrt{1 - |\tau|^2 \left(1 - \frac{\Lambda_{pq}}{\Lambda_{p,q+1}}\right)} \,, \end{aligned} \tag{6.8}$$

where we have introduced the rescaled quantity $\tau := \xi_1 \xi_2 / |J|^2 |t|^2$. The operators (6.8) are correct up to terms of the order $O(|\tau|^2(1 - |\tau|^2)(\xi_1 - \xi_2)^2)$ which vanish if the tachyon is on-shell (see below) and are suppressed for large p, q.

[18] Again we omit the time integral from the definition (4.2).

6.2 Topological nature of the U(1) instanton charge and the tachyon potential

In [18] it was shown that the instanton number (4.14), which is a surface term, in noncommutative space also for the case of U(1) gauge group has a topological origin. The crucial input for this analysis is that the connections ∇_α are of the form as given in our ansatz (6.2). The tachyonic excitation does not contribute to the instanton number, or more exactly to the Pontrjagin number for non-self-dual configurations, since it is located at the origin $z_1 = z_2 = 0$. The shift operator(s) S introduce an axial asymmetry in the sense that they either shift in the first or second copy of the Hilbert space corresponding to the two constituting noncommutative planes characterised by θ_1 and θ_2. As mentioned above the choice of the shift operator (4.9) is gauge dependent. In our case we have chosen the operator which shifts in the second Hilbert space. As a result the gauge field is located around $z_1 = 0$ and becomes a phase at the boundary of the plane parametrised by z_2. After integration over the z_1-plane the Pontrjagin charge reduces to a winding number of the gauge field around S^1 on the z_2-plane and is thus characterised by $\pi_1(\mathrm{U}(1))$ (see [18] for details). Although the analysis in [18] was done for instantons, i.e. self-dual solutions, all arguments go through with our ansatz (6.2) if the operators $Z^\dagger_\alpha$ become $\sim c^\dagger_\alpha$ far away from the centre. In terms of the matrix elements A_{pq} and B_{pq} large distance to the centre means large p, q. Thus the results of [18] are valid for more general cases than self-dual configurations.

Based on the above arguments we can use the famous Bogomolnyi trick to write the potential as a sum of the Pontrjagin number and the anti-self-dual part of the gauge field $F_{\mu\nu}$:

$$V = -\frac{1}{8g^2_{YM}} \int \mathrm{Tr}\left[(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2\right] + \frac{4\pi^2}{g^2_{YM}}\,. \tag{6.9}$$

The unstable and the final state fall in the class of solutions (6.7). If we assume that also for arbitrary $|t|^2$ the solution to (6.5) falls into this class the tachyon potential (6.9) simplifies to:

$$V(|t|) = \frac{4\pi^2}{g^2_{YM}} \frac{\xi_1\xi_2}{2} \left[\left(\frac{|J|^2}{\xi_1\xi_2} - |t|^2\right)^2 + \frac{2}{\xi_1\xi_2}\right]. \tag{6.10}$$

Regarding the dimension of the potential note that in five dimensions the Yang-Mills coupling is no longer dimensionless but has the dimension of a length. Clearly, the extrema of the potential (6.10) are given by

$$t = 0\,, \qquad |t|^2 = |\rho|^2 = \frac{|J|^2}{\xi_1\xi_2}\,, \tag{6.11}$$

where the first one is the unstable extremum and the second one is a stable (local) minimum. Note that in the latter case the tachyon mode (6.1) coincides with the axial term of the self-dual solution (5.10). The local depth of the potential is given by

$$\begin{aligned} \Delta V &= V(0) - V(|\rho|) \\ &= \frac{2\pi^2}{g^2_{YM}} \frac{|J|^4}{\xi_1\xi_2} = \frac{2\pi^2}{g^2_{YM}} \left(\sqrt{\frac{\theta_1}{\theta_2}} - \sqrt{\frac{\theta_2}{\theta_1}}\right)^2 . \end{aligned} \tag{6.12}$$

Thus the depth of the potential exactly matches the mass defect ΔM (4.16) of the $D0$-$D4$ system and is therefore in perfect agreement with Sen's conjecture [46].

The approximate solution (6.8) we found above becomes exact for the extrema (6.11) and coincides at these points with the unstable state (4.13) or the final state (5.10), respectively. When the tachyon is off-shell (6.8) is valid only up to terms of order $O((\xi_1 - \xi_2)^2)$, but this error is suppressed for the matrix elements where q and p are large. Anyway, (6.8) fulfils the requirements discussed above, such that it gives rise to a topological origin of the Pontrjagin number. Thus (6.8) smoothly parametrises a path in field space between the unstable and stable connection (4.13), (5.10) such that for every $\tau \in [0, 1]$ (6.8) is a well defined U(1) connection with constant integer Pontrjagin number one.

6.3 Discussion

The crucial input in our computation of the tachyon potential is that also for the U(1) gauge theory the Pontrjagin number is a topological quantity. This also explains why the unstable state (4.12) which has the Pontrjagin number one (4.14) does not decay into the vacuum but to the BPS state (5.10). In the $2 + 1$-dimensional case as it was considered in [19] no such topological constraint exists. In contrary there exists a continuous path in field space, which connects the unstable solitonic state with the vacuum configuration, see also [47].

Another crucial input is the ansatz (6.2) which is based on symmetry arguments as given in [19]. So the relevant degrees of freedom in the condensation process are operators of the form (6.3). According to our identification of the fluctuation spectrum with string excitations (4.30), (4.33) these are the massless V_{44} strings. This is in contrast to the string field theory calculation of [2]. In this case the zero level truncation approximation for a $(0, 4)$ string field also gives a quartic tachyon potential but the depth of the potential reproduces only 25 percent of the mass defect (4.16) in the considered range of the B-field. In [2] it was supposed that the reason for this is the existence of a large number of low-lying $(0, 4)$-string states. But in our calculation the whole condensation effect comes from the massless $(4, 4)$ strings. It would be nice if a higher level string field theory calculation could resolve this seeming contradiction.

In [48] also a quartic tachyon potential was obtained in a noncommutative field theory calculation. But the potential differs crucially from our result (6.10). It does not give the correct mass defect (4.16). We believe that the reason for this is that in [48] very strong conditions on the integrated out fields were imposed. It is doubtful whether a nontrivial solution to these conditions exists. This would mean that the minimum of the tachyon potential as given in [48] does not really describe a local minimum of the theory.

7. Conclusions: a proposal and a puzzle

We discussed several aspects of the unstable $D0$-$D4$ system with a B-field background in terms of noncommutative super Yang-Mills theory. In particular we studied zero-mode moduli and the fluctuation spectrum. For its stable limit (the small instanton), i.e. when the B-field becomes self-dual, we observed an equivalence of the fluctuation spectrum with the one of the fluxon solution on $\mathbb{R}^2_\theta \times \mathbb{R}$ [3]. We argued that this equivalence of the spectra is due to T-duality but postponed a detailed discussion to future studies.

Secondly we constructed a single U(1) instanton for non-self-dual B-field backgrounds using the ADHM method. We were able to exactly compute the excitation spectrum for this state. It would be interesting if this could be compared with a string (field) theory calculation. The spectrum looks very different from the spectrum of the unstable and small instanton background, not only quantitatively but also qualitatively. Although the classical solutions are continuously connected in the limit of self-dual B-field the spectra are not. The number of states significantly increases at the self-dual point compared to the excitation spectrum of the instanton for non-self-dual B-field. Also additional zero-modes appear at the self-dual point. The scalar field zero-modes associated with translations of the $D0$-brane transverse to the $D4$-brane are not present for the instanton with non-self-dual B-field. Thus the $D0$-brane is confined in the $D4$-brane.

Finally we computed the tachyon potential for the unstable $D0$-$D4$ system. A crucial observation is that the Pontrjagin number in noncommutative $\mathbb{R}^4_\theta$ is topological also for the U(1) gauge theory and is thus conserved during the condensation process. Therefore the unstable $D0$-$D4$ condensates to a BPS state with same topological charge instead to the vacuum. The in this way obtained quartic tachyon potential reproduces the whole mass defect of the formation of the $D0$-$D4$ bound state in the Seiberg-Witten limit. The relevant degrees of freedom in the condensation process are the massless $(4,4)$ strings living on the $D4$-brane. This result is in contrast to string field calculations where only 25 percent of the mass defect could be reproduced and as the relevant degrees of freedom the low-lying $(0,4)$ strings were expected [2].

Based on the observed significant change in the number of excited states when the B-field becomes self-dual we propose the appearance of a phase transition or at least a cross over at this point in the parameter space for the B-field background. This would accompany the deconfinement of the $D0$-brane from the $D4$-brane. At the same time the effective theory on the $D0$-brane changes from the Higgs branch to the Coulomb branch. We hope to address this question in the future.

From a field theoretical point of view a puzzle remains. When considering one-loop instanton corrections a consistent regularisation plays an important role. In [49, 50, 24, 25] such a susy-preserving dimensional regularisation for topological nontrivial backgrounds was developed by embedding the theory in a higher dimensional space with the same field content.[19] When using this regularisation, which also seems to apply to the noncommutative case, only the continuous spectrum contributes to quantum corrections. The contributions of the discrete spectrum cancel each other because of the residual supersymmetry in the BPS background. But if the continuous spectrum is gapless as we obtained it here in all cases it gives only scale-less integrals which vanish in dimensional regularisation. On the other hand, also in noncommutative theories the bare parameters may renormalise non-trivially [7−9]. This especially means that in the $\mathcal{N}=2$ case the divergent renormalisation of the coupling constant would not be cancelled by an analogous contribution from the instanton determinants. The net result would be divergent. A pos-

[19] For an alternative regularisation suited for noncommutative theories see [51].

sible resolution could be that noncommutativity introduces a scale in the spectral density of the massless states. We have to postpone a detailed analysis of this issue to future studies.

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